Assignment 19

Absorption into Stagnant

Governing Eq.

$$D\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial t} = kc \qquad , \quad D = \text{Diffusion Coeff.}$$

$$\begin{cases} t=0 , & c=0 \\ x=0 , & c=c^* \\ x=\infty , & c=0 \end{cases}$$

Let
$$\begin{cases} Y = \frac{c}{c^*} \\ \tau = kt \\ z = \sqrt{\frac{k}{D}}\, x \end{cases}$$

Then,
$$\frac{\partial^2 Y}{\partial z^2} - \frac{\partial Y}{\partial \tau} = Y$$

Taking LT,

$$\frac{d^2\bar{Y}}{dz^2} - s\bar{Y} - \cancelto{0}{Y(z,0)} = \bar{Y}$$

$$\begin{cases} \tau=0, & Y=0, & \bar{Y}=0 \\ z=0, & Y=1 & , \bar{Y}=\frac{1}{s} \\ z=\infty, & Y=0, & \bar{Y}=0 \end{cases}$$

$$\frac{d^2\bar{Y}}{dz^2} - (s+1)\bar{Y} = 0$$

$$\bar{Y} = c_1 e^{-\sqrt{s+1}\, z} + c_2 e^{\sqrt{s+1}\, z}$$

$$z=\infty , \ \bar{Y}=0 \rightarrow c_2 = 0$$

$$\bar{Y} = c_1 e^{-\sqrt{s+1}\, z}$$

$$z=0, \ \bar{Y}=\frac{1}{s} \rightarrow c_1 = \frac{1}{s}$$

$$\boxed{\bar{Y} = \frac{1}{s} e^{-\sqrt{s+1}\, z}}$$

Applying,

Thm: $\mathcal{L}^{-1}\{\frac{1}{s}\phi(s)\} = \int_0^t \mathcal{L}^{-1}\{\phi(s)\}\, dt$

$$Y = \int_0^\tau \mathcal{L}^{-1}\{e^{-\sqrt{s+1}\, z}\}\, d\tau$$

Thm: $\mathcal{L}^{-1}\{\phi(s-a)\} = e^{at}\mathcal{L}^{-1}\{\phi(s)\}$

$$\mathcal{L}^{-1}\{e^{-\sqrt{s+1}\, z}\} = e^{-\tau}\,\underline{\underline{\mathcal{L}^{-1}\{e^{-z\sqrt{s}}\}}}$$

Applying Inverse Formula

$$\mathcal{L}^{-1}\{e^{-\alpha\sqrt{s}}\} = \frac{\alpha}{2\sqrt{\pi t^3}} \exp\left(-\frac{\alpha^2}{4t}\right), \quad \alpha \to z$$

$$\mathcal{L}^{-1}\{e^{-z\sqrt{s}}\} = \frac{z}{2\sqrt{\pi\tau^3}} \exp\left(-\frac{z^2}{4\tau}\right)$$

$$\mathcal{L}^{-1}\{e^{-\sqrt{s+1}\, z}\} = \frac{z e^{-\tau}}{2\sqrt{\pi\tau^3}} \exp\left(-\frac{z^2}{4\tau}\right)$$

$$= \boxed{\frac{z}{2\sqrt{\pi\tau^3}} \exp\left(-\frac{z^2}{4\tau} - \tau\right)}$$

$$Y = \int_0^\tau \frac{z}{2\sqrt{\pi\tau^3}} \exp\left(-\frac{z^2}{4\tau} - \tau\right) d\tau \qquad (A)$$

Let $u = \frac{z}{2\sqrt{\tau}} - \sqrt{\tau} \quad \rightarrow \quad du = \left[\frac{1}{4} z \tau^{-3/2} - \frac{1}{2}\tau^{-1/2}\right] d\tau$

$$u^2 = \frac{z^2}{4\tau} + \tau - z$$

$$Y = \int_0^{\tau} \frac{2}{\sqrt{\pi}} \left(\frac{z}{4} \tau^{-3/2} \overbrace{+ \frac{1}{2} \tau^{-1/2} - \frac{1}{2} \tau^{-1/2}}^{0 \,=} \right) \exp\left(-\frac{z^2}{4\tau} - \tau\right) d\tau , \qquad \text{(B)}$$

note that Eq. A = Eq. B

$$= \int_0^{\tau} \frac{2}{\sqrt{\pi}} \left(\frac{z}{4} \tau^{-3/2} + \frac{1}{2} \tau^{-1/2} \right) \exp\left(-\frac{z^2}{4\tau} - \tau\right) d\tau$$

$$+ \int_0^{\tau} - \frac{\tau^{-1/2}}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4\tau} - \tau\right) d\tau$$

$$\boxed{\begin{aligned} \text{Let} \quad & u = \frac{z}{2\sqrt{\tau}} - \sqrt{\tau} \\ & du = \left[-\frac{1}{4} z \tau^{-3/2} - \frac{1}{2} \tau^{-1/2} \right] d\tau \\ & u^2 = \frac{z^2}{4\tau} + \tau - z \end{aligned}}$$

$$= \int_{\infty}^{u} \frac{2}{\sqrt{\pi}} \left[\frac{\frac{z}{4} \tau^{-3/2} + \frac{1}{2} \tau^{-1/2}}{-\frac{1}{4} z \tau^{-3/2} - \frac{1}{2} \tau^{-1/2}} \right] \exp(-u^2 - z) \, du$$

$$+ \underbrace{\int_0^{\tau} - \frac{\tau^{-1/2}}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4\tau} - \tau\right) d\tau}$$

$$= - \frac{2}{\sqrt{\pi}} \int_{\infty}^{u} \exp(-u^2 - z) \, du \quad + \quad ''$$

$$= - e^{-z} \frac{2}{\sqrt{\pi}} \int_{\infty}^{u} \exp(-u^2) du \quad + \quad ''$$

$$\int_{\infty}^{u} = - \int_{u}^{\infty} = - \int_0^{\infty} + \int_0^{u}$$

$$= - e^{-z} \frac{2}{\sqrt{\pi}} \left[- \int_0^{\infty} \exp(-u^2) du + \int_0^{u} \exp(-u^2) du \right] + \ ''$$

$$= e^{-z}\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \int_0^u \exp(-u^2)\,du\right] + \quad ''$$

$$= e^{-z}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^u \exp(-u^2)\,du\right] + \quad ''$$

$$Y = e^{-z}\,\mathrm{erfc}(u) + \int_0^\tau -\frac{\tau^{-1/2}}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4\tau}-\tau\right)d\tau \quad , \quad u = \frac{z}{2\sqrt{\tau}} - \sqrt{\tau}$$

From Eq (A). Y is an odd function against z

$$Y(z,\tau) = -Y(-z,\tau)$$

$$Y(z,\tau) = e^{-z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{\tau}} - \sqrt{\tau}\right) + \int_0^\tau -\frac{\tau^{-1/2}}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4\tau}-\tau\right)d\tau$$

$$-)\quad -Y(z,\tau) = Y(-z,\tau) = e^{z}\,\mathrm{erfc}\left(\frac{-z}{2\sqrt{\tau}} - \sqrt{\tau}\right) + \quad ''$$

$$2Y(z,\tau) = e^{-z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{z}\,\mathrm{erfc}\left(\frac{-z}{2\sqrt{\tau}} - \sqrt{\tau}\right)$$

$$\| $$

$$e^{z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{\tau}} + \sqrt{\tau}\right)$$

$$[\mathrm{erfc}(x) = \mathrm{erfc}(-x)]$$

$$\boxed{Y(z,\tau) = \frac{1}{2}\left[e^{-z}\,\mathrm{erf}\left(\frac{z}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{z}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right]}$$

where $z = \sqrt{\frac{k}{D}}\,x, \quad \tau = kt$

Eq. 20C 3-2, P 653, BSL

Typo in the Book, $\begin{pmatrix} + \rightarrow - \\ \mathrm{erf} \rightarrow \mathrm{erfc} \end{pmatrix}$