$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sin(x) \tag{3.190}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cos(x) \tag{3.191}$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sinh(x) \tag{3.192}$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cosh(x) \tag{3.193}$$

$$\frac{d}{dx}\left[x^p Z_p(\lambda x)\right] = \left\{\begin{array}{ll} \lambda x^p Z_{p-1}(\lambda x), & Z = J, Y, I \\ -\lambda x^p Z_{p-1}(\lambda x), & Z = K \end{array}\right\} \tag{3.194}$$

$$\frac{d}{dx}\left[x^{-p} Z_p(\lambda x)\right] = \left\{\begin{array}{ll} -\lambda x^{-p} Z_{p+1}(\lambda x), & Z = J, Y, K \\ \lambda x^{-p} Z_{p+1}(\lambda x), & Z = I \end{array}\right\} \tag{3.195}$$

$$\frac{d}{dx}\left[Z_p(\lambda x)\right] = \left\{\begin{array}{ll} \lambda Z_{p-1}(\lambda x) - \frac{p}{x} Z_p(\lambda x), & Z = J, Y, I \\ -\lambda Z_{p-1}(\lambda x) - \frac{p}{x} Z_p(\lambda x), & Z = K \end{array}\right\} \tag{3.196}$$

By applying the recurrence relations, these can also be written in the more useful form

$$\frac{d}{dx}\left[Z_p(\lambda x)\right] = \left\{\begin{array}{ll} -\lambda Z_{p+1}(\lambda x) + \frac{p}{x} Z_p(\lambda x), & Z = J, Y, K \\ \lambda Z_{p+1}(\lambda x) + \frac{p}{x} Z_p(\lambda x), & Z = I \end{array}\right\} \tag{3.197}$$

$$Z_p(\lambda x) = \frac{\lambda x}{2p}\left[Z_{p+1}(\lambda x) + Z_{p-1}(\lambda x)\right], \qquad Z = J, Y \tag{3.198}$$

$$I_p(\lambda x) = \frac{-\lambda x}{2p}\left[I_{p+1}(\lambda x) - I_{p-1}(\lambda x)\right] \tag{3.199}$$

$$K_p(\lambda x) = \frac{\lambda x}{2p}\left[K_{p+1}(\lambda x) - K_{p-1}(\lambda x)\right] \tag{3.200}$$

$$J_{-n}(\lambda x) = (-1)^n J_n(\lambda x) \tag{3.201}$$

$$I_{-n}(\lambda x) = I_n(\lambda x) \tag{3.202}$$

$$K_{-n}(\lambda x) = K_n(\lambda x) \tag{3.203}$$

$$\int \lambda x^p J_{p-1}(\lambda x)\, dx = x^p J_p(\lambda x) \tag{3.204}$$

and

$$\int \lambda x^p I_{p-1}(\lambda x)\, dx = x^p I_p(\lambda x) \tag{3.205}$$

$$\int_0^x J_k(\lambda\xi) J_k(\beta\xi)\xi\, d\xi = \frac{x}{\lambda^2 - \beta^2}\left[\lambda J_k(\lambda x)\cdot J_{k+1}(\beta x) - \beta J_k(\lambda x) J_{k+1}(\beta x)\right] \tag{3.206}$$

and if $\lambda = \beta$, this gives the useful result

$$\int_0^x \left[J_k(\lambda\xi)\right]^2 \xi\, d\xi = \frac{1}{2}x^2\left[J_k^2(\lambda x) - J_{k-1}(\lambda x) J_{k+1}(\lambda x)\right] \tag{3.207}$$