< Proof of S-L Theorem >

Let y_m, y_n be solutions of S-L problem for λ_m, λ_n respectively.

Then

$$\begin{cases} \dfrac{d(py_m')}{dx} + (q + \lambda_m^2 r) y_m = 0 & \text{— (1)} \\ \dfrac{d(py_n')}{dx} + (q + \lambda_n^2 r) y_n = 0 & \text{— (2)} \end{cases}$$

$(1) \times y_n - (2) \times y_m$,

$$y_n \frac{d}{dx}(p y_m') + \lambda_m^2 r y_n y_m - y_m \frac{d}{dx}(p y_n') - \lambda_n^2 r y_n y_m = 0$$

or

$$(\lambda_m^2 - \lambda_n^2) r y_m y_n = y_m \frac{d}{dx}(p y_n') - y_n \frac{d}{dx}(p y_m').$$

Integrate from a to b,

$$(\lambda_m^2 - \lambda_n^2) \int_a^b r y_m y_n \, dx = \underbrace{\int_a^b y_m \frac{d}{dx}(p y_n') dx}_{(A)} - \underbrace{\int_a^b y_n \frac{d}{dx}(p y_m') dx}_{(B)}$$

$$(A) = p y_m y_n' \Big|_a^b - \int_a^b p y_n' y_m' \, dx$$

$$\left(\begin{array}{ll} u = y_m & dv = d(p y_n') dx \\ du = y_m' dx & v = p y_n' \end{array} \right.$$

Likewise

$$(B) = p y_n y_m' \Big|_a^b - \int_a^b p y_m' y_n' \, dx$$

Therefore,

$$(\lambda_m^2 - \lambda_n^2) \int_a^b r y_m y_n \, dx = p y_m y_n' \Big|_a^b - p y_n y_m' \Big|_a^b = p(y_m y_n' - y_n y_m') \Big|_a^b$$

$$= p(b) \{ y_m(b) y_n'(b) - y_n(b) y_m'(b) \}$$

$$- p(a) \{ y_m(a) y_n'(a) - y_n(a) y_m'(a) \}$$

From B.C.

$$y_n'(a) = \frac{a_1}{a_2} y_n(a), \quad y_m'(a) = \frac{a_1}{a_2} y_m(a)$$

$$y_n'(b) = \frac{b_1}{b_2} y_n(b), \quad y_m'(b) = \frac{b_1}{b_2} y_m(b)$$

(over)

$$(\lambda_m^2 - \lambda_n^2) \int_a^b r\, y_m y_n \, dx$$

$$= p(b)\left\{ \frac{b_1}{b_2} y_m(b) y_n(b) - \frac{b_1}{b_2} y_n(b) y_m(b) \right\} - p(a)\left\{ \frac{a_1}{a_2} y_m(a) y_n(a) - \frac{a_1}{a_2} y_n(a) y_m(a) \right\}$$

$$= 0$$

$\therefore \int_a^b r\, y_m y_n \, dx = 0$ or $\{y_i\}$ are orthogonal w.r.t $r(x)$

Note: If a_2 or b_2 is zero, substitute $y_n(a) = \frac{a_2}{a_1} y_n'(a)$

Thus a_1 or a_2, b_1 or b_2 can be zero.