International Journal of Cognitive Computing in Engineering 2 (2021) 191–195

Contents lists available at ScienceDirect

International Journal of Cognitive Computing in Engineering

journal homepage: https://www.keaipublishing.com/en/journals/international-journal-of-cognitive-computing-in-engineering/

Closed-form solution for the max-min fairness problem in non-orthogonal multiple access systems

Shiyong Yang[a], Shiwen Mao[b], Bin Tan[c,*]

[a] School of Electronic and Information Engineering, Jiujiang University, Jiujiang, Jiangxi Province 332005, China
[b] Department of Electrical and Computer Engineering, Auburn University, Auburn, AL 36849-5201, USA
[c] College of Electronics and Information Engineering, Jinggangshan University, Ji'an, Jiangxi Province 343009, China

ARTICLE INFO

Keywords:
Max-min fairness
Closed-form solution
Non-orthogonal multiple access
Power allocation

ABSTRACT

Non-orthogonal multiple access (NOMA) can improve both the spectrum efficiency and the number of users in wireless communication systems. In the downlink of NOMA, the base station superimposes multiple data flows in the power domain and the users decode the information using successive interference cancellation SIC. The performance of downlink NOMA systems is highly dependent on the power allocation scheme for all users. This paper investigates the max-min fairness problem that maximizes the minimum achievable user rate, aiming to ensure fairness for all users in downlink NOMA systems. Although the max-min fairness problem in downlink NOMA systems is non-convex, we obtain its closed-form optimal solution in this paper. This result will be useful for the analysis and design of future NOMA systems. Numerical results validate the correctness and effectiveness of our closed-form optimal solution.

1. Introduction

Non-orthogonal multiple access (NOMA) has emerged as a key technology for the fifth generation (5G) and beyond wireless communication systems (Dai et al., 2015; Ding et al., 2016; Islam et al., 2018; Maddikunta et al., to appear). NOMA has recently received significant attention since it enables the multiplexing of multiple users' data on the same time and frequency resource, which improves system spectral efficiency (Dai et al., 2015; Islam et al., 2018; Xiao et al., 2018). The basic idea of NOMA is that the base station serves multiple users in the same channel resource block (e.g., same time and frequency). In the downlink NOMA system, the base station superimposes the signals of different users using superposition coding with an appropriate power allocation in the power domain, and the receivers exploit successive interference cancellation (SIC) to distinguish each other's messages and remove the multi-user interference. NOMA can improve the spectrum efficiency and support a larger number of users in the wireless system.

Fairness is one of the most important performance metrics in downlink NOMA systems. Fairness can be achieved through appropriate power allocation of the superimposed, transmitted signals among all users. The performance of downlink NOMA systems is highly dependent on the power allocation scheme for all users (Wang et al., 2016). Max-min fairness is a common performance measure widely adopted in prior works, which is to maximize the minimum achievable user rate to achieve fairness among all users.

Researchers have investigated the fairness problem of the simple case of two users in downlink NOMA systems, such as max-min rate proportional fairness (Choi, 2016), max-min fairness and proportional fairness (Zhu et al., 2017), and the optimal throughput fairness tradeoff (Xing et al., 2018). A sub-optimal solution for the joint optimization of beamforming and max-min fairness in downlink NOMA systems were proposed in Xiao et al. (2019). To ensure fairness of multiple users in downlink NOMA systems, Timotheou and Krikidis (2015) proposed a bisection-based iterative algorithm to obtain the optimal solution to the max-min fairness problem. The algorithm proposed in Timotheou and Krikidis (2015) was implemented with the bisection procedure and requires an unknown number of iterations with a high complexity, which limits its application in practical situations.

This paper investigates the problem of max-min fairness among all users in downlink NOMA systems. Although the max-min fairness problem is non-convex, we provide a problem formulation and successfully obtain the optimal solution in its closed-form. To the best of our knowledge, there is no such closed-form optimal solution available in the prior literature. The derived closed-form solution to the max-min fairness problem of downlink NOMA systems can help to quickly and efficiently allocate the optimal power among all users to maximize the minimum achievable user rate. It is suitable and useful for analysis and design of future practical NOMA systems. Our contributions in this paper are summarized as follows. We first provide the closed-form optimal solution to the max-min fairness problem of downlink NOMA systems. The

* Corresponding author.
E-mail address: tanbin@jgsu.edu.cn (B. Tan).

https://doi.org/10.1016/j.ijcce.2021.11.001
Received 19 August 2021; Received in revised form 11 November 2021; Accepted 18 November 2021
Available online 20 November 2021

derived closed-form optimal solution is efficient and easy to be implemented in practical situations. We then provide a simulation study and the numerical results validate our analysis.

The rest of this paper is organized as follows. We present the system model and problem formulation in Section 2. The closed-form solution is derived and proven in Section 3. Numerical results are presented to demonstrate the effectiveness of the closed-form solution in Section 4. Finally, conclusions are drawn in Section 5.

2. System model and problem formulation

We consider a downlink NOMA system where a base station (BS) serves K users, denoted by $\mathcal{U}_k$, $k \in \mathcal{K} = \{1, 2, \ldots, K\}$. Both the BS and the users are assumed to be equipped with a single antenna. The signal transmitted by the BS can be expressed as

$$x = \sum_{k=1}^{K} \sqrt{\beta_k P} s_k, \tag{1}$$

where s_k is the symbol of user $\mathcal{U}_k$, P is the total transmit power, and β_k is the fraction of total power allocated to user $\mathcal{U}_k$. The β_k's satisfy $\sum_{k=1}^{K} \beta_k \leq 1$. The received signal at user $\mathcal{U}_k$ is given by

$$y_k = \sum_{k=1}^{K} \sqrt{\beta_k P} h_k s_k + z_k, \tag{2}$$

where h_k is channel coefficient from the BS to user $\mathcal{U}_k$, and $z_k \sim \mathcal{CN}(0, \sigma_k^2)$ is the additive white Gaussian noise.

Each user applies SIC to decode its signal from the mix. Define $N_k = \sigma_k^2 / |h_k|^2$, $k = 1, 2, \ldots, K$. Without loss of generality, we assume that

$$N_1 < N_2 < \ldots < N_K, \tag{3}$$

i.e., $\mathcal{U}_1$ is the strongest user and $\mathcal{U}_K$ is the weakest user. Thus, $\mathcal{U}_k$ is able to first decode the signals of all users $\mathcal{U}_i$ for $i > k$, and the remove them from its received signal, and treats the signals from all users $\mathcal{U}_j$ for $j < k$ as interference. Therefore, the signal to interference-plus-noise ratio of $\mathcal{U}_k$ using SIC is written as

$$\eta_k = \frac{\beta_k P}{\sum_{i=1}^{k-1} \beta_i P + N_k}. \tag{4}$$

And the data rate of $\mathcal{U}_k$ is given by

$$R_k(\beta_1, \ldots, \beta_K) = \frac{1}{2} \log\left(1 + \frac{\beta_k P}{\sum_{i=1}^{k-1} \beta_i P + N_k}\right), \; k = 1, 2, \ldots, K. \tag{5}$$

The performance of the NOMA system relies on the power allocation among all users. In this paper, we investigate the optimization of power allocation to ensure max-min fairness, which is to maximize the minimum achievable user rate. The max-min fairness problem is formulated as follows.

$$\begin{aligned} \max_{\beta_1, \beta_2, \ldots, \beta_K} \min_{k \in \mathcal{K}} \quad & R_k(\beta_1, \beta_2, \ldots, \beta_K) \\ \text{s.t.} \quad & \sum_{k=1}^{K} \beta_k \leq 1 \\ & \beta_k \geq 0, \; \forall k \in \mathcal{K}. \end{aligned} \tag{6}$$

Problem (6) is non-convex, and hence is hard to solve directly using standard optimization solvers. Timotheou and Krikidis (2015) proposed a polynomial-time algorithm to obtain the optimal solution by transforming problem (6) into a sequence of linear programs. However, the algorithm proposed in Timotheou and Krikidis (2015) is based on the bisection procedure and requires an unknown number of iterations. In this paper, we derive the closed-form optimal solution for the max-min fairness problem (6).

3. Closed-form solution

The closed-form optimal solution to problem (6) is given in the following theorem.

Theorem 1. *The optimal solution to the max-min fairness problem (6) is given by*

$$\beta_k^* = \frac{\theta N_k}{P} + \frac{1}{P} \sum_{j=1}^{k-1} \sum_{i=1}^{j} C_{k-i-1}^{j-i} N_i \theta^{k-j+1}, \; k = 1, 2, \ldots, K, \tag{7}$$

where $C_n^r = \frac{n!}{r!(n-r)!}$ and θ is the real positive root of the following equation of degree K with the unknown X:

$$\sum_{k=1}^{K} \left(\sum_{j=1}^{k} C_{K-j}^{k-j} N_j \right) X^{K-k+1} - P = 0. \tag{8}$$

Proof. In order to prove Theorem 1, we first provide a lemma.

Lemma 1. *The optimal solution to problem (6) must be obtained at $\sum_{k=1}^{K} \beta_k = 1$.*

The proof of Lemma 1 is given in the Appendix. According to Lemma 1, problem (6) can be rewritten as

$$\begin{aligned} \max_{\beta_1, \beta_2, \ldots, \beta_K} \min_{k \in \mathcal{K}} \quad & R_k(\beta_1, \beta_2, \ldots, \beta_K) \\ \text{s.t.} \quad & \sum_{k=1}^{K} \beta_k = 1, \\ & \beta_k \geq 0, \; \forall k \in \mathcal{K}. \end{aligned} \tag{9}$$

It follows (5) that

$$\beta_k = \left(2^{R_k} - 1\right) \left(\sum_{i=1}^{k-1} \beta_i + \frac{N_k}{P} \right), \; k = 1, 2, \ldots, K. \tag{10}$$

For problem (9), it is obvious that the optimal solution will be obtained at

$$R_1(\beta_1, \ldots, \beta_K) = R_2(\beta_1, \ldots, \beta_K) = \ldots = R_K(\beta_1, \ldots, \beta_K). \tag{11}$$

Let $X = e^{2R_1} - 1$. Following (11), (10) can be rewritten as

$$\beta_k = \sum_{i=1}^{k-1} X \beta_i + \frac{X N_k}{P}, \; k = 1, 2, \ldots, K. \tag{12}$$

Then, we have

$$\beta_k = \sum_{j=1}^{k-1} \left(N_j \left(\sum_{i=1}^{k-j} C_{k-j-1}^{i-1} X^{k-j-i+2} \right) \right) + \frac{X N_k}{P}, \; k = 1, 2, \ldots, K. \tag{13}$$

Due to the fact that $\sum_{k=1}^{K} \beta_k = 1$, we have

$$\sum_{k=1}^{K} \left(\sum_{j=1}^{k-1} \left(\sum_{i=1}^{j} C_{k-i-1}^{j-i} N_i \right) X^{k-j+1} + X N_k \right) = P. \tag{14}$$

$$\sum_{k=1}^{K} \left(\sum_{j=1}^{k} C_{K-j}^{k-j} N_j \right) X^{K-k+1} - P = 0. \tag{15}$$

The solution $X = e^{2R_1} - 1$ is optimal to problem (15), which is an equation of degree K with unknown X. But problem (15) has K solutions that may either be positive or negative, real or complex. According to the constrain $\beta_k \geq 0$, $\forall k \in \mathcal{K}$, X must be real and positive. Fortunately, the coefficients of the unknown X in (15) are

$$a_k = \sum_{j=1}^{k} C_{K-j}^{k-j} N_j > 0, \; k = 1, 2, \ldots, K. \tag{16}$$

The number of changes of signs in the sequence $\{a_1, a_2, \ldots, a_K, -P\}$ of the coefficients of polynomial

$$\left(\sum_{k=1}^{K} \left(\sum_{j=1}^{k} C_{K-j}^{k-j} N_j \right) X^{K-k+1} - P \right)$$

Table 1
Simulations parameter setting.

Parameter	Value
Cell radius	1000 m
Minimum distance form user to BS	35 m
Carrier frequency	2 GHz
Path loss model	$128.1 + 37.6\log_{10} d$ dB, where d is in km
Shadowing	Log-normal, 10 dB standard deviation
Fading	Rayleigh fading with variance 1
Noise power spectral density	−174 dbm/Hz
System bandwidth W	5 MHz
Number of users K	5, 20
Total power P	10 W

is equal to 1. According to the Descartes Theorem (Mignotte, 1992), problem (15) has only one real positive root. Denote the real positive root of (15) as θ. Substituting θ into (13), we can obtain the optimal solution to problem (9) as follows.

$$\beta_k^* = \frac{\theta N_k}{P} + \frac{1}{P}\sum_{j=1}^{k-1}\sum_{i=1}^{j} C_{k-i-1}^{j-i} N_i \theta^{k-j+1},\ k = 1, 2, \ldots, K. \tag{17}$$

□

4. Numerical results

In this section, we present the numerical results to demonstrate the effectiveness of the derived closed-form solution to the max-min fairness problem of downlink NOMA systems. We consider a hexagonal cell of diameter 1000 m, with one base station located at its center and K users distributed uniformly at random in the cell. The system bandwidth is $W = 5$ MHz. The radio propagation model including path loss, shadowing, and Rayleigh fading, and the parameters are the same as in Lou et al. (2020). The simulation parameters and channel model are summarized in Table 1. We compare the closed-form solution with the proposed algorithm in Timotheou and Krikidis (2015).

We evaluate the closed-form solution and the proposed algorithm in Timotheou and Krikidis (2015) in our simulations with a system of $K = 5$ users. The max-min rate of the 5 users is $R_1 = R_2 = \ldots = R_5 = 8,278,940$ bps, which is calculated directly by the closed-form solution. The max-min rate of the 5 users obtained by the proposed algorithm in Timotheou and Krikidis (2015) is also $R_1 = R_2 = \ldots = R_5 = 8,278,940$ bps. But the proposed algorithm in Timotheou and Krikidis (2015) obtained the same result with 27 iterations. When the number of users is increased to $K = 20$, the max-min rate of the 20 users is $R_1 = R_2 = \ldots = R_20 = 2,819,367$ bps as calculated directly by the closed-form solution. The proposed algorithm in Timotheou and Krikidis (2015) obtains the same result after 31 iterations. The closed-solution can compute the same optimal solution but at a greatly reduced complexity as compared with the benchmark scheme.

5. Conclusions

This paper investigated the max-min fairness problem in a downlink NOMA system. Although the max-min fairness problem is non-convex, we were able to obtain the optimal solution in its closed form. The closed-form solution of the max-min fairness problem of downlink NOMA systems allows quick and efficient allocation of the optimal power among users to maximize the minimum achievable user rate, and is suitable and useful for the analysis and design of practical communication systems. Numerical results validate the correctness and effectiveness of the closed-form optimal solution.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported in part by the National Nature Science Foundation of China under Grant 61762053, and the State Scholarship Fund of the China Scholarship Council under 201908360042. S. Mao's work is supported in part by the NSF under Grant ECCS-1923717.

Appendix A. Proof of Lemma 1

Proof. We use Proof by Contradiction in this proof. Suppose that the optimal solution to problem (6) is obtained at $\beta_1 = \bar{\beta}_1, \ldots, \beta_K = \bar{\beta}_K$ and $\sum_{k=1}^{K} \bar{\beta}_k < 1$. Denote the corresponding user rates as $\bar{R}_1(\bar{\beta}_1, \ldots, \bar{\beta}_K), \ldots, \bar{R}_K(\bar{\beta}_1, \ldots, \bar{\beta}_K)$. Let $\alpha = 1 - \sum_{k=1}^{K} \bar{\beta}_k$ and we have $\alpha > 0$. Suppose that the maximum of minimum in $\{\bar{R}_1(\bar{\beta}_1, \ldots, \bar{\beta}_K), \ldots, \bar{R}_K(\bar{\beta}_1, \ldots, \bar{\beta}_K)\}$ is $\bar{R}_l(\bar{\beta}_1, \ldots, \bar{\beta}_K)$. Now we prove that there always exists a new power allocation scheme $\{\gamma_1, \ldots, \gamma_K\}$ with $\gamma_k \geq 0$, for all k, and $\sum_{k=1}^{K} \gamma_k = 1$, which makes $R_l(\gamma_1, \ldots, \gamma_K) > \bar{R}_l(\bar{\beta}_1, \ldots, \bar{\beta}_K)$ and $R_k(\gamma_1, \ldots, \gamma_K) = \bar{R}_k(\bar{\beta}_1, \ldots, \bar{\beta}_K)$, for all $k \neq l$.

There are two cases that need to be considered.

CASE I $\bar{R}_l(\bar{\beta}_1, \ldots, \bar{\beta}_K)$ is the only minimum rate in $\{\bar{R}_1(\bar{\beta}_1, \ldots, \bar{\beta}_K), \ldots, \bar{R}_K(\bar{\beta}_1, \ldots, \bar{\beta}_K)\}$, i.e., $\bar{R}_k(\bar{\beta}_1, \ldots, \bar{\beta}_K) > \bar{R}_l(\bar{\beta}_1, \ldots, \bar{\beta}_K)$, for all $k \neq l$.

If $l = K$, let $\gamma_l = \gamma_K = \bar{\beta}_K + \alpha$ and $\gamma_k = \bar{\beta}_k$, for all $k \neq l$. Then we have

$$R_k(\gamma_1, \ldots, \gamma_K) = \bar{R}_k(\bar{\beta}_1, \ldots, \bar{\beta}_K),\ \forall k \neq l \tag{18}$$

$$\begin{aligned} R_K(\gamma_1, \ldots, \gamma_K) &= \tfrac{1}{2}\log\left(1 + \frac{(\bar{\beta}_K+\alpha)P}{\sum_{i=1}^{K-1}\bar{\beta}_i P + N_K}\right) \\ &> \tfrac{1}{2}\log\left(1 + \frac{\bar{\beta}_K P}{\sum_{i=1}^{K-1}\bar{\beta}_i P + N_K}\right) \\ &= \bar{R}_K(\bar{\beta}_1, \ldots, \bar{\beta}_K). \end{aligned} \tag{19}$$

This is a contradiction to that $\bar{R}_l(\bar{\beta}_1, \ldots, \bar{\beta}_K)$ is the maximum of minimum in $\{\bar{R}_1(\bar{\beta}_1, \ldots, \bar{\beta}_K), \ldots, \bar{R}_K(\bar{\beta}_1, \ldots, \bar{\beta}_K)\}$.

If $l < K$, set

$$\xi = \frac{\alpha}{\prod_{m=1}^{K-l}\left\{\sum_{i=1}^{l+m}\bar{\beta}_i P + N_{l+m}\right\}\left\{\sum_{i=1}^{l+m-1}\bar{\beta}_i P + N_{l+m}\right\}^{-1}}, \tag{20}$$

$$\gamma_k = \bar{\beta}_k,\ k = 1, 2, \ldots, l-1, \tag{21}$$

$$\gamma_l = \bar{\beta}_l + \xi, \tag{22}$$

$$\gamma_{l+n} = \bar{\beta}_{l+n} + \xi\left(\frac{\bar{\beta}_{l+n}P}{\sum_{i=1}^{l+n-1}\bar{\beta}_i P + N_{l+n}}\right)\left(\prod_{m=1}^{n-1}\frac{\sum_{i=1}^{l+m}\bar{\beta}_i P + N_{l+m}}{\sum_{i=1}^{l+m-1}\bar{\beta}_i P + N_{l+m}}\right),\ n = 1, 2, \ldots, K-l. \tag{23}$$

Then, we have

$$R_k(\gamma_1, \ldots, \gamma_K) = \bar{R}_k(\bar{\beta}_1, \ldots, \bar{\beta}_K),\ k = 1, 2, \ldots, l-1, \tag{24}$$

$$R_l(\gamma_1, \ldots, \gamma_K) = \frac{1}{2}\log\left(1 + \frac{(\bar{\beta}_l + \xi)P}{\sum_{i=1}^{l-1}\bar{\beta}_i P + N_l}\right)$$

$$> \frac{1}{2}\log\left(1+\frac{\bar{\beta}_l P}{\sum_{i=1}^{l-1}\bar{\beta}_i P + N_l}\right) = \bar{R}_l(\bar{\beta}_1,\dots,\bar{\beta}_K), \tag{25}$$

$$\begin{aligned} &R_{l+n}(\gamma_1,\dots,\gamma_K)\\ =&\frac{1}{2}\log\left(1+\frac{\bar{\beta}_{l+n}P+\xi P\left(\frac{\bar{\beta}_{l+n}P}{\sum\limits_{i=1}^{l+n-1}\bar{\beta}_i P+N_{l+n}}\right)\left(\prod\limits_{m=1}^{n-1}\frac{\sum\limits_{i=1}^{l+m}\bar{\beta}_i P+N_{l+m}}{\sum\limits_{i=1}^{l+m-1}\bar{\beta}_i P+N_{l+m}}\right)}{\sum\limits_{i=1}^{l+n-1}\bar{\beta}_i P+N_{l+n}+\xi P+\xi P\sum\limits_{t=1}^{n-1}\left(\frac{\bar{\beta}_{l+t}P}{\sum\limits_{i=1}^{l+t-1}\bar{\beta}_i P+N_{l+t}}\right)\left(\prod\limits_{m=1}^{t-1}\frac{\sum\limits_{i=1}^{l+m}\bar{\beta}_i P+N_{l+m}}{\sum\limits_{i=1}^{l+m-1}\bar{\beta}_i P+N_{l+m}}\right)}\right)\\ =&\bar{R}_{l+n}(\bar{\beta}_1,\dots,\bar{\beta}_K),\ n=1,2,\dots,K-l. \end{aligned} \tag{26}$$

This is a contradiction to that $\bar{R}_l(\bar{\beta}_1,\dots,\bar{\beta}_K)$ is the maximum of minimum in $\{\bar{R}_1(\bar{\beta}_1,\dots,\bar{\beta}_K),\dots,\bar{R}_K(\bar{\beta}_1,\dots,\bar{\beta}_K)\}$.

CASE II There are L $(L>1)$ equal minimums in $\{\bar{R}_1(\bar{\beta}_1,\dots,\bar{\beta}_K),\dots,\bar{R}_K(\bar{\beta}_1,\dots,\bar{\beta}_K)\}$. Let the L equal minimums in $\{\bar{R}_1(\bar{\beta}_1,\dots,\bar{\beta}_K),\dots,\bar{R}_K(\bar{\beta}_1,\dots,\bar{\beta}_K)\}$ be $\bar{R}_{l_1}(\bar{\beta}_1,\dots,\bar{\beta}_K)=\dots=\bar{R}_{l_L}(\bar{\beta}_1,\dots,\bar{\beta}_K)$ and $l_1<\dots<l_L$. Let $\lambda=\alpha/L$ and set

$$\mu_1=\frac{\lambda}{\prod\limits_{m=1}^{K-l_1}\left\{\sum\limits_{i=1}^{l_1+m}\bar{\beta}_i P+N_{l_1+m}\right\}\left\{\sum\limits_{i=1}^{l_1+m-1}\bar{\beta}_i P+N_{l_1+m}\right\}^{-1}}, \tag{27}$$

$$\theta_k^1=\bar{\beta}_k,\ k=1,2,\dots,l_1-1, \tag{28}$$

$$\theta_{l_1}^1=\bar{\beta}_{l_1}+\mu_1, \tag{29}$$

$$\theta_{l_1+n}^1=\bar{\beta}_{l_1+n}+\mu_1\left(\frac{\bar{\beta}_{l_1+n}P}{\sum\limits_{i=1}^{l_1+n-1}\bar{\beta}_i P+N_{l_1+n}}\right)\left(\prod\limits_{m=1}^{n-1}\frac{\sum\limits_{i=1}^{l_1+m}\bar{\beta}_i P+N_{l_1+m}}{\sum\limits_{i=1}^{l_1+m-1}\bar{\beta}_i P+N_{l_1+m}}\right), \tag{30}$$

$$n=1,2,\dots,K-l_1. \tag{30}$$

Then we have

$$R_k(\theta_1^1,\dots,\theta_K^1)=\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ k=1,2,\dots,l_1-1, \tag{31}$$

$$R_{l_1}(\theta_1^1,\dots,\theta_K^1)=\frac{1}{2}\log\left(1+\frac{(\bar{\beta}_{l_1}+\mu_1)P}{\sum_{i=1}^{l_1-1}\bar{\beta}_i P+N_{l_1}}\right) > \bar{R}_{l_1}(\bar{\beta}_1,\dots,\bar{\beta}_K), \tag{32}$$

$$\begin{aligned} &R_{l_1+n}(\theta_1^1,\dots,\theta_K^1)\\ =&\frac{1}{2}\log\left(1+\frac{\bar{\beta}_{l_1+n}P+\mu_1 P\left(\frac{\bar{\beta}_{l_1+n}P}{\sum\limits_{i=1}^{l_1+n-1}\bar{\beta}_i P+N_{l_1+n}}\right)\left(\prod\limits_{m=1}^{n-1}\frac{\sum\limits_{i=1}^{l_1+m}\bar{\beta}_i P+N_{l_1+m}}{\sum\limits_{i=1}^{l_1+m-1}\bar{\beta}_i P+N_{l_1+m}}\right)}{\sum\limits_{i=1}^{l_1+n-1}\bar{\beta}_i P+N_{l_1+n}+\mu_1 P+\mu_1 P\sum\limits_{t=1}^{n-1}\left(\frac{\bar{\beta}_{l_1+t}P}{\sum\limits_{i=1}^{l_1+t-1}\bar{\beta}_i P+N_{l_1+t}}\right)\left(\prod\limits_{m=1}^{t-1}\frac{\sum\limits_{i=1}^{l_1+m}\bar{\beta}_i P+N_{l_1+m}}{\sum\limits_{i=1}^{l_1+m-1}\bar{\beta}_i P+N_{l_1+m}}\right)}\right)\\ =&\bar{R}_{l_1+n}(\bar{\beta}_1,\dots,\bar{\beta}_K),\ n=1,2,\dots,K-l_1. \end{aligned} \tag{33}$$

For $s=2,3,\dots,L$, set

$$\mu_s=\frac{\lambda}{\prod\limits_{m=1}^{K-l_s}\left\{\sum\limits_{i=1}^{l_s+m}\theta_i^1 P+N_{l_s+m}\right\}\left\{\sum\limits_{i=1}^{l_s+m-1}\theta_i^{s-1}P+N_{l_s+m}\right\}^{-1}}, \tag{34}$$

$$\theta_k^s=\theta_k^{s-1},\ k=1,2,\dots,l_s-1, \tag{35}$$

$$\theta_{l_s}^s=\theta_{l_{s-1}}^{s-1}+\mu_s, \tag{36}$$

$$\theta_{l_s+n}^s=\theta_{l_{s-1}+n}^{s-1}+\mu_s\left(\frac{\theta_{l_s+n}^{s-1}P}{\sum\limits_{i=1}^{l_s+n-1}\theta_i^{s-1}P+N_{l_s+n}}\right)\left(\prod\limits_{m=1}^{n-1}\frac{\sum\limits_{i=1}^{l_s+m}\theta_i^{s-1}P+N_{l_s+m}}{\sum\limits_{i=1}^{l_s+m-1}\theta_i^{s-1}P+N_{l_s+m}}\right),$$

$$n=1,2,\dots,K-l_s, \tag{37}$$

Then we have

$$R_k(\theta_1^L,\dots,\theta_K^L)=\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ k=1,2,\dots,l_1-1, \tag{38}$$

$$R_{l_1}(\theta_1^L,\dots,\theta_K^L)=\frac{1}{2}\log\left(1+\frac{(\bar{\beta}_{l_1}+\mu_1)P}{\sum_{i=1}^{l_1-1}\bar{\beta}_i P+N_{l_1}}\right) > \bar{R}_{l_1}(\bar{\beta}_1,\dots,\bar{\beta}_K), \tag{39}$$

$$R_k(\theta_1^L,\dots,\theta_K^L)=\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ l_1<k<l_2, \tag{40}$$

$$R_{l_2}(\theta_1^L,\dots,\theta_K^L)=\frac{1}{2}\log\left(1+\frac{(\theta_{l_1}^1+\mu_2)P}{\sum_{i=1}^{l_2-1}\theta_i^2 P+N_{l_2}}\right) > \bar{R}_{l_2}(\bar{\beta}_1,\dots,\bar{\beta}_K), \tag{41}$$

$$R_k(\theta_1^L,\dots,\theta_K^L)=\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ l_{L-1}<k<l_L, \tag{42}$$

$$R_{l_K}(\theta_1^L,\dots,\theta_K^L)=\frac{1}{2}\log\left(1+\frac{(\theta_{l_{L-1}}^{L-1}+\mu_L)P}{\sum_{i=1}^{l_L-1}\theta_i^L P+N_{l_L}}\right) > \bar{R}_{l_K}(\bar{\beta}_1,\dots,\bar{\beta}_K), \tag{43}$$

…

$$R_{l_K+n}(\theta_1^L,\dots,\theta_K^L)=\frac{1}{2}\log\left(1+\frac{\theta_{l_L+n}^L P}{\sum_{i=1}^{l_L+n-1}\theta_i^L P+N_{l_L+n}}\right) = \bar{R}_{l_L+n}(\bar{\beta}_1,\dots,\bar{\beta}_K),\ n=1,2,\dots,K-l_L, \tag{44}$$

$$\theta_k^L\ge 0,\ k=1,2,\dots,K, \tag{45}$$

$$\sum_{k=1}^K\theta_k^L=1. \tag{46}$$

It follows that

$$R_k(\theta_1^L,\dots,\theta_K^L)=\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ \forall k\ne l_1,l_2,\dots,l_L, \tag{47}$$

$$R_k(\theta_1^L,\dots,\theta_K^L)>\bar{R}_k(\bar{\beta}_1,\dots,\bar{\beta}_K),\ \forall k=l_1,l_2,\dots,l_L, \tag{48}$$

$$\theta_k^L\ge 0,\ k=1,2,\dots,K, \tag{49}$$

$$\sum_{k=1}^K\theta_k^L=1. \tag{50}$$

This is a contradiction to that $\bar{R}_{l_1}(\bar{\beta}_1,\dots,\bar{\beta}_K)=\dots=\bar{R}_{l_L}(\bar{\beta}_1,\dots,\bar{\beta}_K)$ is the maximum of minimum in $\{\bar{R}_1(\bar{\beta}_1,\dots,\bar{\beta}_K),\dots,\bar{R}_K(\bar{\beta}_1,\dots,\bar{\beta}_K)\}$.

To sum up, the optimal solution to problem (6) must be obtained at $\sum_{k=1}^K\beta_k=1$. □

References

Choi, J. (2016). Power allocation for max-sum rate and max-min rate proportional fairness in NOMA. *IEEE Communications Letters, 20*, 2055–2058.

Dai, L., Wang, B., Yuan, Y., Han, S., Chih-Lin, I., & Wang, Z. (2015). Non-orthogonal multiple access for 5G: Solutions, challenges, opportunities, and future research trends. *IEEE Communications Magazine, 53*, 74–81.

Ding, Z., Fan, P., & Poor, H. V. (2016). Impact of user pairing on 5G nonorthogonal multiple-access downlink transmissions. *IEEE Transactions on Vehicular Technology, 65*, 6010–6023.

Islam, S. M. R., Zeng, M., Dobre, O. A., & Kwak, K. (2018). Resource allocation for NOMA downlink systems: Key techniques and open issues. *IEEE Wireless Communications, 25*, 40–47.

Lou, S., Coupechoux, M., & Chen, C. S. (2020). Joint subcarrier and power allocation in NOMA: Optimal and approximate algorithms. *IEEE Transactions on Signal Processing, 68*, 2215–2230.

Maddikunta, P. K. R., Pham, Q.-V., Prabadevi, B., Deepa, N., Dev, K., & Gadekallu, T. R. et al. Industry 5.0: A survey on enabling technologies and potential applications. Elsevier Journal of Industrial Information Integration, to appear.

Mignotte, M. (1992). *Mathematics for computer algebra*. Berlin, Germany: Springer-Verlag.

Timotheou, S., & Krikidis, I. (2015). Fairness for non-orthogonal multiple access in 5Gsystems. *IEEE Signal Processing Letters, 22*, 1647–1651.

Wang, C., Chen, J., & Chen, Y. (2016). Power allocation for a downlink non-orthogonal multiple access system. *IEEE Wireless Communications Letter, 5*, 532–535.

Xiao, Z., Zhu, L., Choi, J., Xia, P., & Xia, X. G. (2018). Joint power allocation and beamforming for non-orthogonal multiple access (NOMA) in 5G millimeter wave communications. *IEEE Transactions on Wireless Communications, 17*, 2961–2974.

Xiao, Z., Zhu, L., Gao, Z., Wu, D. O., & Xia, X. (2019). User fairness non-orthogonal multiple access (NOMA) for millimeter-wave communications with analog beamforming. *IEEE Transactions on Wireless Communications, 18*, 3411–3423.

Xing, H., Liu, Y., Nallanathan, A., Ding, Z., & Poor, H. V. (2018). Optimal throughput fairness tradeoffs for downlink non-orthogonal multiple access over fading channels. *IEEE Transactions on Wireless Communications, 17*, 3556–3571.

Zhu, J., Wang, J., Huang, Y., He, S., You, X., & Yang, L. (2017). On optimal power allocation for downlink non-orthogonal multiple access systems. *IEEE Journal on Selected Areas in Communications, 35*, 2744–2757.