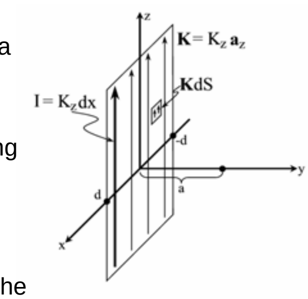
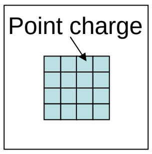
## Magnetostatics – Surface Current Density

A *sheet current*, **K** (A/m²) is considered to flow in an infinitesimally thin layer.




Method 1: The surface charge problem can be treated as a sheet consisting of a continuous point charge distribution.

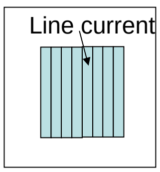



The *Biot-Savart law* can also be written in terms of surface current density by replacing Id**L** with **K** dS

$$\mathbf{H} = \int \frac{\mathbf{K}dS \times \mathbf{a}_R}{4\pi R^2}$$

Important Note: The sheet current's direction is given by the vector quantity **K** rather than by a vector direction for dS.

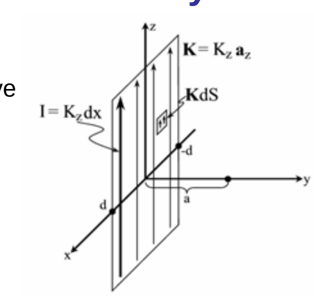
Method 2: The surface current sheet problem can be treated as a sheet consisting of a continuous series of line currents.




$$d\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \frac{K_z dz}{2\pi\rho}\mathbf{a}_\phi$$

## Magnetostatics – Surface Current Density

Method 2: Example

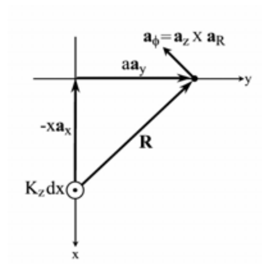



Example 3.4: We wish to find **H** at a point centered above an infinite length ribbon of sheet current

We can treat the ribbon as a collection of infinite length lines of current $K_z$dx.

Each line of current will contribute d**H** of field given by

$$d\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$




The differential segment I=$K_z$dx

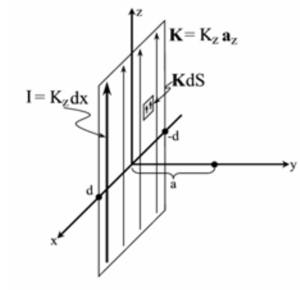
The vector drawn from the source to the test point is

$$\mathbf{R} = -x\mathbf{a}_x + a\mathbf{a}_y$$

Magnitude: $\rho = \mathrm{R} = \sqrt{x^2 + a^2}$ Unit Vector: $\mathbf{a}_z \times \mathbf{a}_R = \mathbf{a}_\phi$

## Magnetostatics – Surface Current Density

To find the total field, we integrate from x = -d to x = +d




$$\mathbf{H} = \int_{-d}^{d} \frac{K_z dx\left(\mathbf{a}_z \times \left(-x\mathbf{a}_x + a\mathbf{a}_y\right)\right)}{2\pi\left(x^2 + a^2\right)}$$

$$\mathbf{H} = \frac{K_z}{2\pi}\left[\cancel{\int_{-d}^{d} \frac{-xdx\mathbf{a}_y}{x^2 + a^2}} - \int_{-d}^{d} \frac{adx\mathbf{a}_x}{x^2 + a^2}\right] = \frac{-K_z a}{2\pi}\mathbf{a}_x \boxed{\left[\int_{-d}^{d} \frac{dx}{x^2 + a^2}\right]}$$

We notice by symmetry arguments that the first term inside the brackets, the $\mathbf{a}_y$ component, is zero.

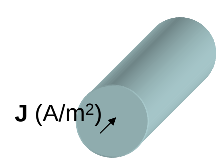
The second integral can be evaluated using the formula given in Appendix D.

$$\int_{-d}^{+d} \frac{dx}{\left(x^2 + a^2\right)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Bigg|_{-d}^{+d} = \frac{1}{a}\left(\tan^{-1}\left(\frac{d}{a}\right) - \tan^{-1}\left(\frac{-d}{a}\right)\right) = \frac{2}{a}\tan^{-1}\left(\frac{d}{a}\right)$$

$$\mathbf{H} = \frac{-K_z}{\pi}\tan^{-1}\left(\frac{d}{a}\right)\mathbf{a}_x \quad \xrightarrow{\text{Infinite Current Sheet}} \quad \mathbf{H} = -\frac{K_z}{2}\mathbf{a}_x.$$

## Magnetostatics – Volume Current Density

Current and Current Densities:
*Linear current* I (A)
*Surface current density* **K** (A/m)
*Volume current* density **J** ($A/m^2$)




The *Biot-Savart law* can also be written in terms of volume current density by replacing Id**L** with **J**dv

$$\mathbf{H} = \int \frac{\mathbf{J}dv \times \mathbf{a}_R}{4\pi R^2}$$

For many problems involving surface current densities, and indeed for most problems involving volume current densities, solving for the magnetic field intensity using the *Law of Biot-Savart* can be quite cumbersome and require numerical integration.

For most problems that we will encounter with volume charge densities, we will have sufficient symmetry to be able to solve for the fields using *Ampere's Circuital Law* (Next topic).

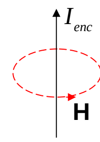
## Magnetostatics – Ampere's Circuital Law

In electrostatics problems that featured a lot of symmetry we were able to apply Gauss's Law to solve for the electric field intensity much more easily than applying Coulomb's Law.

Likewise, in magnetostatic problems with sufficient symmetry we can employ *Ampere's Circuital Law* more easily than the Law of Biot-Savart.

Ampere's Circuital Law says that the integration of **H** around any closed path is equal to the net current enclosed by that path.

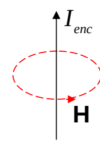
$$\oint \mathbf{H} \bullet d\mathbf{L} = I_{enc}$$




The line integral of **H** around a closed path is termed the *circulation* of **H**.

## Magnetostatics – Ampere's Circuital Law

Amperian Path: Typically, the *Amperian Path* (analogous to a Gaussian Surface) is chosen such that the integration path is either tangential or normal to **H** (over which H is constant) .




(i) Tangential to **H**, i.e.

$$\vec{H} \bullet d\vec{L} = HdL\cos\theta = HdL\cos 0° = HdL$$

(ii) Normal to **H**, i.e.

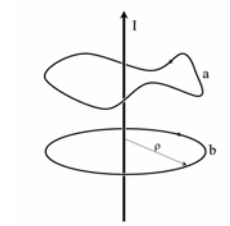
$$\vec{H} \bullet d\vec{L} = HdL\cos\theta = HdL\cos 90° = 0$$

The direction of the circulation is chosen such that the right hand rule is satisfied. That is, with the thumb in the direction of the current, the fingers will curl in the direction of the circulation.

## Magnetostatics – Ampere's Circuital Law
## Application to Line Current

<u>Example 3.5:</u> Here we want to find the magnetic field intensity everywhere resulting from an infinite length line of current situated on the z-axis .

The figure also shows a pair of Amperian paths, *a* and *b*. Performing the circulation of **H** about either path will result in the same current I. But we choose path *b* that has a constant value of *Hf* around the circle specified by the radius $\rho$.




In the Ampere's Circuital Law equation, we substitute **H** = $H_\phi \mathbf{a}_\phi$ and d**L** = $\rho d\phi \mathbf{a}_\phi$, or

$$\oint \mathbf{H} \bullet d\mathbf{L} = I_{enc}$$

$$= \int_0^{2\pi} H_\phi \mathbf{a}_\phi \bullet \rho d\phi \mathbf{a}_\phi = 2\pi\rho H_\phi = I$$

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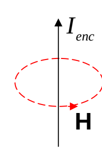


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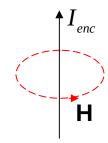
## Magnetostatics – Ampere's Circuital Law

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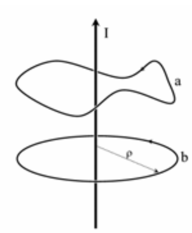



The figure also shows a pair of Amperian paths, *a* and *b*. Performing the circulation of **H** about either path will result in the same current I. But we choose path *b* that has a constant value of *Hf* around the circle specified by the radius *ρ*.

In the Ampere's Circuital Law equation, we substitute **H** = $H_\phi \mathbf{a}_\phi$ and d**L =** $\rho d\phi \mathbf{a}_\phi$, or

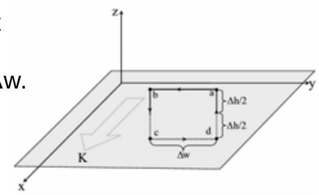
$$\oint \mathbf{H} \bullet d\mathbf{L} = I_{enc}$$

$$= \int_0^{2\pi} H_\phi \mathbf{a}_\phi \bullet \rho d\phi \mathbf{a}_\phi = 2\pi\rho H_\phi = I$$

## Magnetostatics – Ampere's Circuital Law Application to Current Sheet

<u>Example 3.6:</u> Let us now use Ampere's Circuital Law to find the magnetic field intensity resulting from an infinite extent sheet of current.

Let us consider a current sheet with uniform current density $\mathbf{K} = K_x \mathbf{a}_x$ in the z = 0 plane along with a rectangular Amperian Path of height $\Delta$h and width $\Delta$w.




Amperian Path: In accordance with the right hand rule, where the thumb of the right hand points in the direction of the current and the fingers curl in the direction of the field, we'll perform the circulation in the order

$$a \to b \to c \to d \to a.$$

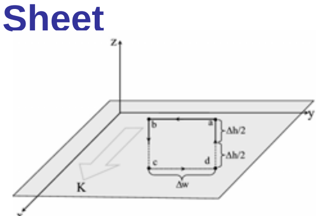
*Ampere's Circuital Law*

$$\oint \mathbf{H} \bullet d\mathbf{L} = I_{enc}$$

$$= \int_a^b \mathbf{H} \bullet d\mathbf{L} + \int_b^c \mathbf{H} \bullet d\mathbf{L} + \int_c^d \mathbf{H} \bullet d\mathbf{L} + \int_d^a \mathbf{H} \bullet d\mathbf{L}.$$

## Magnetostatics – Ampere's Circuital Law Application to Current Sheet

From symmetry arguments, we know that there is no $H_x$ component.




$$\oint \mathbf{H} \bullet d\mathbf{L} = \int_a^b \mathbf{H} \bullet d\mathbf{L} + \cancel{\int_b^c \mathbf{H} \bullet d\mathbf{L}} + \int_c^d \mathbf{H} \bullet d\mathbf{L} + \cancel{\int_d^a \mathbf{H} \bullet d\mathbf{L}}$$

Above the sheet $\mathbf{H} = H_y(-\mathbf{a}_y)$ and below the sheet $\mathbf{H} = H_y \mathbf{a}_y$.

$$\oint \mathbf{H} \bullet d\mathbf{L} = \int_{\Delta w}^{0} H_y \left(-\mathbf{a}_y\right) \bullet dy\mathbf{a}_y + \int_0^{\Delta w} H_y \mathbf{a}_y \bullet dy\mathbf{a}_y$$

$$= 2H_y \Delta w$$

The current enclosed by the path is just $I = \int_0^{\Delta w} K_x dy = K_x \Delta w$

Equating the above two terms gives $H_y = \frac{K_x}{2}$

A general equation for an infinite current sheet: $\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N$

where $\mathbf{a}_N$ is a normal vector from the sheet current to the test point.

## Magnetostatics – Curl and the Point Form of Ampere's Circuital Law

In electrostatics, the *concept of divergence* was employed to find the point form of Gauss's Law from the integral form. A non-zero divergence of the electric field indicates the presence of a charge at that point.

In magnetostatics, *curl* is employed to find the point form of Ampere's Circuital Law from the integral form. A non-zero curl of the magnetic field will indicate the presence of a current at that point.

To begin, let's apply Ampere's Circuital Law to a path surrounding a small surface. Dividing both sides by the small surface area, we have the circulation per unit area

$$\frac{\oint \mathbf{H}\bullet d\mathbf{L}}{\Delta S} = \frac{I_{enc}}{\Delta S} \xrightarrow{\text{Taking the limit}} \lim_{\Delta S \to 0} \frac{\oint \mathbf{H}\bullet d\mathbf{L}}{\Delta S} = \lim_{\Delta S \to 0} \frac{I_{enc}}{\Delta S}$$

If the direction of the $\Delta$S vector is chosen in the direction of the current, $\mathbf{a}_n$. Multiplying both sides of by $\mathbf{a}_n$ we have

$$\lim_{\Delta S \to 0} \frac{\oint \mathbf{H}\bullet d\mathbf{L}}{\Delta S}\mathbf{a}_n = \lim_{\Delta S \to 0} \frac{I_{enc}}{\Delta S}\mathbf{a}_n$$

## Magnetostatics – Curl and the Point Form of Ampere's Circuital Law

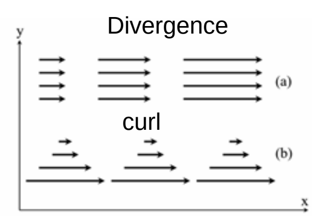
$$\boxed{\lim_{\Delta S \to 0} \frac{\oint \mathbf{H}\bullet d\mathbf{L}}{\Delta S}\mathbf{a}_n} = \boxed{\lim_{\Delta S \to 0} \frac{I_{enc}}{\Delta S}\mathbf{a}_n}$$

$$\boxed{\text{curl } \mathbf{H} = \nabla \times \mathbf{H}} = \boxed{\mathbf{J}}$$

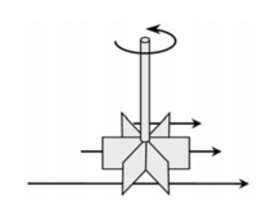
The left side is *the maximum circulation of **H** per unit area as the area shrinks to zero*, called *the curl of **H*** for short. The right side is the current density, **J**

So we now have the *point form* of Ampere's Circuital Law: $\nabla \times \mathbf{H} = \mathbf{J}$

Field lines



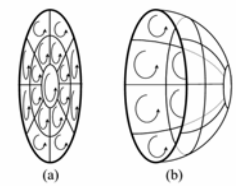

A Skilling Wheel used to measure curl of the velocity field in flowing water



# Magnetostatics – Stokes Theorem

We can re-write Ampere's Circuital Law in terms of a current density as

$$\oint \mathbf{H} \bullet d\mathbf{L} = \int \mathbf{J} \bullet d\mathbf{S}$$




We use the point form of Ampere's Circuital Law to replace J with $\nabla \times \mathbf{H}$

$$\boxed{\oint \mathbf{H} \bullet d\mathbf{L} = \int \left(\nabla \times \mathbf{H}\right) \bullet d\mathbf{S}}.$$

This expression, relating a closed line integral to a surface integral, is known as *Stokes's Theorem* (after British Mathematician and Physicist Sir George Stokes, 1819-1903).

Now suppose we consider that the surface bounded by the contour in Figure (a) is actually a rubber sheet.  In Figure (b), we can distort the surface while keeping it intact.  As long as the surface remains unbroken, Stokes's theorem is still valid!