TOPICS

1) LAPLACE TRANSFORM
2) TRANSFER FUNCTIONS

LAPLACE TRANSFORM

- USED TO SOLVE LINEAR TIME INVARIANT DIFFERENTIAL EQUATIONS
- CONVERT CALCULUS TO ALGEBRA

TABLE OF TRANSFORM PAIRS IN TEXT (PALM)
FRONT COVER OR TABLE 2.2.1 p 39

TABLE OF TRANSFORM PROPERTIES
TABLE 2.2.2 p 42

$\mathcal{L}\{\cdot\}$

TIME DOMAIN

$\rightarrow \dot{x} + 5x = F$

INDEPENDENT VARIABLE → t
DEPENDENT VARIABLE → x

LAPLACE DOMAIN

$sX(s) + 5X(s) = F(s)$

INDEPENDENT VARIABLE → s
DEPENDENT VARIABLE → X

$s = \sigma \pm j\omega$

IN GENERAL

$$\rightarrow \mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

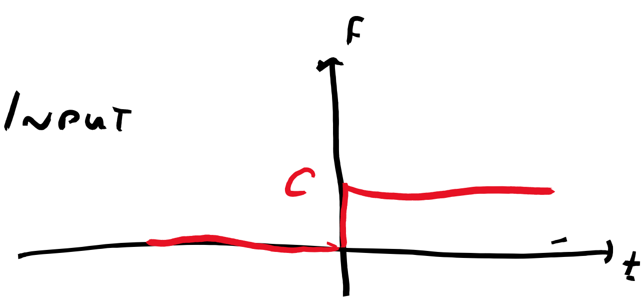
DEFINITION

$$\mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt = X(s)$$

<u>Ex</u>

$$f(t) = \begin{cases} C & t \geq 0 \\ 0 & t < 0 \end{cases}$$

STEP INPUT



$$\mathcal{L}\{f(t)\} = \int_0^{\infty} c\, e^{-st}\, dt$$

$$= c \int_0^{\infty} e^{-st}\, dt$$

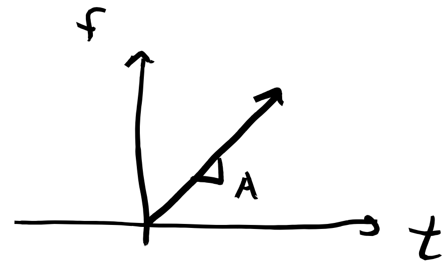
$$= \frac{c}{s} \int_0^{\infty} e^{-u}\, du$$

$$= \frac{c}{s}\left[\overset{1}{\cancel{e^{-0}}} - \overset{0}{\cancel{e^{-\infty}}}\right]$$

$$\boxed{F(s) = \frac{c}{s}}$$

Ex

Ramp Input



$$f(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty A t e^{-st} \, dt$$

$$\boxed{F(s) = \frac{A}{s^2}}$$

Properties of Laplace Transform

1) $\mathcal{L}[a f(t)] = a\mathcal{L}[f(t)]$

2) $\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$

3) Derivative Property

$$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0)$$

Initial condition in time

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sx(0) - \dot{x}(0)$$

Initial conditions

$$\mathcal{L}\left[\frac{d^nx}{dt^n}\right] = s^nX(s) - s^{n-1}x(0) - s^{n-2}\frac{dx}{dt} - \dots$$

4) INTEGRAL PROPERTY

$$\mathcal{L}\left[\int x(t)\,dt\right] = \frac{X(s)}{s} - \frac{g(0)}{s}$$

$$g(0) = \int x(t)\,dt \Big|_{t=0}$$

5) FINAL VALUE THEOREM

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$$

6) INITIAL VALUE THEOREM

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s X(s)$$

Transfer Functions

- Ratio of Output over Input in Laplace Domain
- Ratio of Polynomials in "s"
- LTI ONLY
- Diff. Eq. Can Be Reconstructed From TF
- Inherently Assumes I.C. = 0
- Defines Relationship Between 1 Input and 1 Output SISO

- Denominator = 0 is the Characteristic Equation
- Roots of Numerator "Zeros"
- Roots of Denominator "Poles"

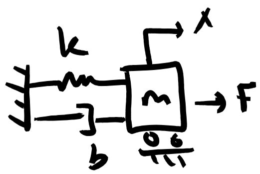
$$TF = \frac{\text{Output}}{\text{Input}}$$

Given $f(t)$ - input

$x(t)$ - output

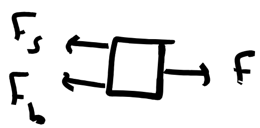
$$\boxed{G(s) = \frac{X(s)}{F(s)}}$$

Ex



Find T.F.

Apply Final Theorem assuming $f(t)$ is unit step input



$$\mathcal{L}\{m\ddot{x} + b\dot{x} + kx = f\}$$

$$\frac{X(s)}{F(s)} = ?$$

$$x_{ss} = C = \frac{f}{k} = \frac{1}{k}$$

$$\mathcal{L}\{m\ddot{x}\} + \mathcal{L}\{b\dot{x}\} + \mathcal{L}\{kx\} = \mathcal{L}\{f\}$$

$$m\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + k\mathcal{L}\{x\} = \mathcal{L}\{f\}$$

$$m\left(s^2X(s) + \cancel{s\dot{x}(0)} + \cancel{\dot{x}(0)}\right) + b\left(sX(s) + \cancel{\dot{x}(0)}\right) + kX(s) = F(s)$$

$ms^2 X(s) + bsX(s) + kX(s) = F(s)$

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{ms^2+bs+k}}$$

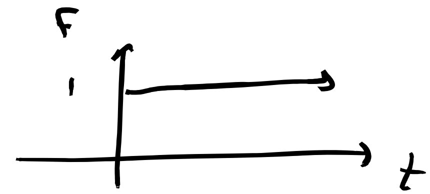
$ms^2+bs+k=0$

$$\lim_{t\to 0} x(t) = \lim_{s\to 0} sX(s)$$

$$= \lim_{s\to 0} s\left(\frac{1}{ms^2+bs+k}\overset{?}{F(s)}\right)$$

$$= \lim_{s\to 0} \cancel{s}\left(\frac{1}{ms^2+bs+k}\frac{1}{\cancel{s}}\right) = \frac{1}{k}$$

$f(t)$ UNIT STEP

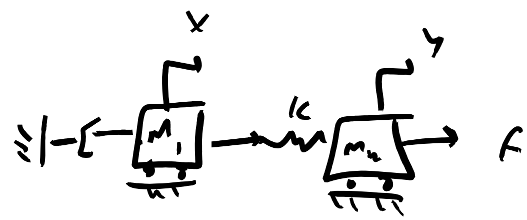


$$\mathcal{L}\{f(t)\} = \frac{1}{s}$$

$$X(s) = \frac{1}{ms^2+bs+k} F(s)$$

$$\underline{\underline{x(t)}} = \mathcal{L}^{-1}\left\{ \frac{1}{ms^2+bs+k} F(s) \right\}$$

$$\begin{cases} m_1\ddot{x} + b\dot{x} + kx = ky \\ m_2\ddot{y} + ky = kx + F \end{cases}$$



How many eigenvalues?

How might I find them?