AUBURN UNIVERSITY

Modeling Electrical-Mechanical Systems (DC Motors)

MECH 3140
Lecture #

Class Update

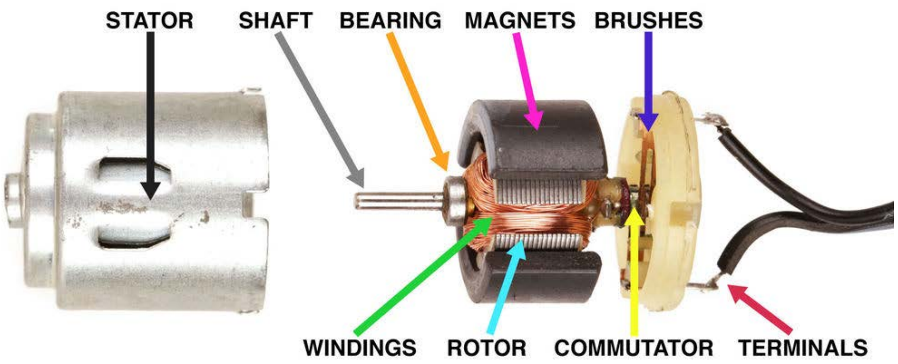
- We will start grading Exam #2 this week
  - It will probably take a couple of weeks as we try and stay on top of other on-line items
- HW #9 is out and contains the last matlab assignment you will have to do
  - There will be a HW #10 which will be problems to prepare for the final exam
- We should be posting the final project this week
  - It will be due the last week of classes and teams will present their power-point slides remotely

Multi-DOF Modeling

- We are starting to looking at modeling multi-DOF systems
- One example will be the DC electric motor.
- This will also be our first example of modeling a combined electrical-mechanical system
  – Note that solenoids and other systems are very analogous

DC Electric Motors

- This is one of the examples where we get to practice modeling Step #0
  - We have to determine which modeling elements to include
  - Here is a picture of the outside and inside of a typical DC motor. Which of our modeling elements do you think we need to use?



“General” Fundamental Operation

- DC motors work by generating opposing magnetic fields that produce a force (or torque) on the shaft.
- In a permanent magnet motor, the opposing field is generated by “pushing” current through a set of coils in the permanent magnetic field
  - The change in “opposition” is created by changing the set of coils that are magnetized (powered) in the armature through the split commutator.

Inputs and Outputs

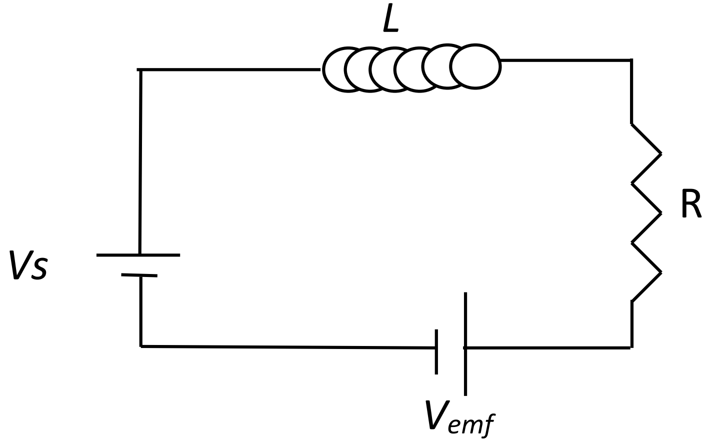
- Lets start by defining the input and output
  - Input could be voltage or current. We will use voltage (but will see later why many in robotics use a current source or current input)
  - The output of the “system” is the position or speed of the motor shaft
    - We will use angular velocity as the fundamental output

Step #0: Electrical Side

- Lets start with modeling the electrical side
- What components do you think we need?
  - Voltage supply (since this is our input)
  - Resistance
    - Why? There is no resistor.
    - The wire inside the motor certainly has resistance (energy loss) that needs to be modeled
  - Inductance
    - Same reasons as above, it is not possible to instantly generate current through all of the wire inside the motor.
  - Voltage Output
    - This is known as the back emf (electromotive force). A voltage is generated in the coils as the travel through the magnetic field
      - This is how a generator works (it's a motor driven in reverse).

Step #0: Electrical Side

- So putting our components together in an electrical schematic to model we get:
  - 1 DOF (I)
  - 1st Order



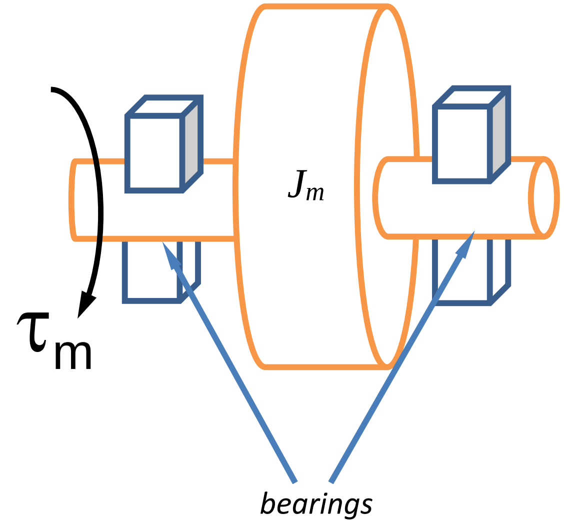
  - Notice this schematic is similar to a previous homework problem where Vemf was a disturbance voltage!
    - This is where that problem came from.

Step #0: Mechanical Side

- Now to the mechanical side
- What components do you think we need?
  - Torque on the shaft (this is what cause the motor to spin)
  - Damping
    - Why? The bearings are not perfect and certainly would have some energy loss that is proportional to velocity
  - Friction
    - There probably is some constant friction force the motor must overcome, but we will ignore this
      - We will assume good bearings or that this force is negligible
  - Inertia
    - For sure the armature has inertia that has to be accounted for.
  - Shaft compliance
    - Possibly, but we will ignore it here (assume the armature is rigid)

Step #0: Mechanical Side

- So putting our components together in an mechanical schematic to model we get:
  - 1 DOF (ω)
  - 1st Order

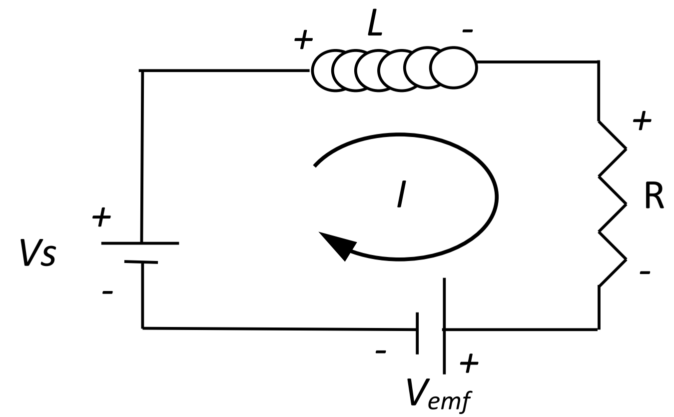


Modeling Step #1 (DOF and Order)

- How many Degrees of Freedom is the *system*?
  - Its 2 DOF: Current and Angular Velocity
    - These are not directly relatable (if I know one, I don't necessarily know the other)
    - This means I should expect 2 coupled differential equations
- What is the *total system* order?
  - Its 2^{nd} Order
    - There are two independent storage elements
      - Inductor
      - Armature Inertia

Step #2 Electrical Side

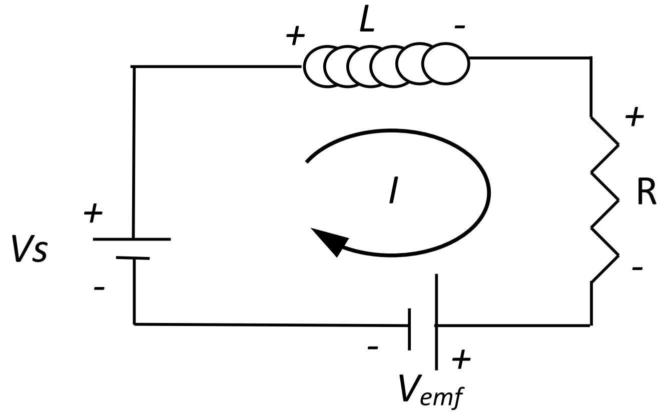
- Assigning current loop and the voltage drops:



- Defining the voltage drops:
  - $V_{emf} = K_b \omega$
    - Back emf is proportional to the motor speed.
    - Kb is a motor constant
  - $V_R = RI$
  - $V_L = L\dot{I}$

Step #3 Electrical Side

- Using Kirchoff's Voltage Law:



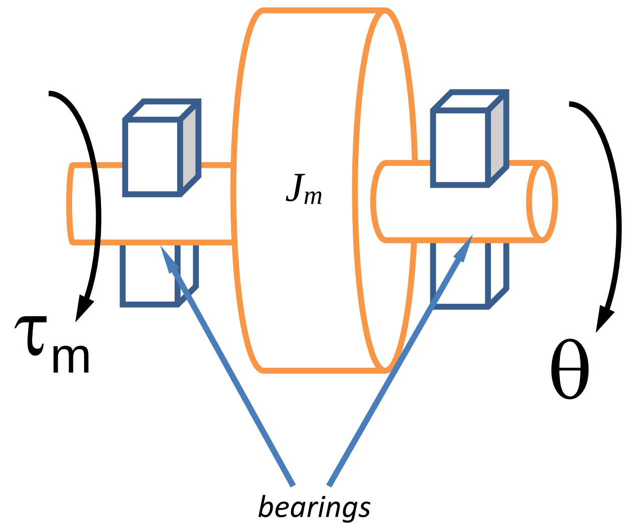
$$\sum V_{Loop} = 0$$

$$-V_s + V_L + V_R + V_{emf} = 0$$

$$L\dot{I} + RI = V_s - K_b\omega$$

Step #2 & #3: Mechanical Side

- Assigning our coordinate (θ)
- Drawing our FBD



  - $\tau_b = b\omega$
  - $\tau_m = K_I I$
    - The motor torque is proportional to motor current
    - This is why robotics use current sources
      - Want to directly input motor torque
    - K_I is a motor constant
      - $K_I = K_b$ (in SI units)

Step #4: Mechanical Side

- Using Newton's Law to sum moments:

$$\sum M = J\ddot{\theta} = J\dot{\omega}$$

$$J\dot{\omega} = -\tau_b + \tau_m$$
$$J\dot{\omega} = -b\omega + K_I I$$

$$J\dot{\omega} + b\omega = K_I I$$

Final Equations of Motion

- We get 2 EOMs
  - Recall it was 2 DOF
    - 1 EOM per DOF
- The equations are coupled
  - The speed differential equation has current
  - The current different equation has speed
- The system is two 1st order coupled differential equations
  - So the system order is 2nd order!

$$J\dot{\omega} + b\omega = K_I I$$

$$L\dot{I} + RI = V_s - K_b\omega$$

Now What?

- How do we solve or analyze?
  - We need….the eigenvalues (of course)
  - So how do we get them?
    - We can't just set the "right side" to zero
      - This would set outputs to zero!
  - Must use our new tools!
    - Transfer Function
    - State Space

Transfer Function

- Take the Laplace of the two LTI equations and setting initial conditions to zero (since we are solving for the transfer function):

$$J\omega(s)s + b\omega(s) = K_I I(s)$$

$$LI(s)s + RI(s) = V_s(s) - K_b\omega(s)$$

- Now What?
  - We have one input, but two outputs (I, ω)
    - Remember a Transfer Function is between a single input and a single output
  - We can not just set the output we don't want to zero – This would change the system!
    - You can set additional inputs to zero

Transfer Function

- Since we have 2 equations, we can solve one for one output and substitute into the other equation

$$J\omega(s)s + b\omega(s) = K_I I(s)$$

$$LI(s)s + RI(s) = V_s(s) - K_b\omega(s)$$

- Which transfer function do we want? $\frac{I(s)}{V_S(s)}$ or $\frac{\omega(s)}{V_S(s)}$
  - For the eigenvalues, it doesn't matter – both transfer functions will have the EXACT same denominator (i.e. same characteristic equation, same eigenvalues)
  - Only the numerator will be different
  - Depends on which output we are interested in
    - I if we want the current (maybe since torque is proportional to I).
    - ω if we want the motor speed

Transfer Function: $\omega(s)/V_s(s)$

- First, lets collect terms:

$$\omega(s)(Js+b) = K_I I(s)$$

$$I(s)(Ls+R) = V_s(s) - K_b\omega(s)$$

- Then use your favorite algebraic trip to eliminate I(s).
  - I am going to solve the 2nd equation for I(s) and substitute into the 1st equation:

$$\omega(s)(Js+b) = K_I\left(\frac{V_s(s) - K_b\omega(s)}{Ls+R}\right)$$

  - Then cleaning up:

$$\omega(s)(Js+b)(Ls+R) + K_I K_b \omega(s) = K_I V_s(s)$$

  - Then solving for the transfer function:

$$\frac{\omega(s)}{V_s(s)} = \frac{K_I}{LJs^2 + (JR+Lb)s + (K_I K_b + Rb)}$$

Transfer Function: $\omega(s)/V_s(s)$

- Notice the denominator is 2nd order!

$$\frac{\omega(s)}{V_s(s)} = \frac{K_I}{LJs^2 + (JR + Lb)s + (K_I K_b + Rb)}$$

- The system characteristic equation is:

$$LJs^2 + (JR + Lb)s + (K_I K_b + Rb) = 0$$

  - So there are two eigenvalues
  - You can use your 2nd order tools!
    - Do you think it is over damped or under damped?

- You can also see the DC Gain is:

$$\frac{K_I}{(K_I K_b + Rb)}$$

State Space

- The other method to find the eigenvalues from multi-DOF systems is state space
  – The eigenvalues are the eigenvalues of the A Matrix

$$J\dot{\omega} + b\omega = K_I I$$

$$L\dot{I} + RI = V_s - K_b\omega$$

- Placing the EOMs into state space representation:
  – Note that the equations were already 1st order (so they were almost already essentially in the correct form)

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R/L & -K_b/L \\ K_I/J & -b/J \end{bmatrix} \begin{bmatrix} I \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} V_s$$

Eigenvalues (State Space)

- Now give the A matrix, we can find the characteristic equation (and eigenvalues)

$$A = \begin{bmatrix} -R/L & -K_b/L \\ K_I/J & -b/J \end{bmatrix}$$

$$eig(A) \Rightarrow \det(sI - A) = 0$$

  - Or in matlab: >>eig(A)

$$\det\left\{sI - \begin{bmatrix} -R/L & -K_b/L \\ K_I/J & -b/J \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} s + R/L & K_b/L \\ -K_I/J & s + b/J \end{bmatrix}\right\} = 0$$

$$\left(s + \frac{R}{L}\right)\left(s + \frac{b}{J}\right) + \frac{K_b K_I}{LJ} = 0$$

Eigenvalues (State Space)

$$s^2 + \left(\frac{R}{L} + \frac{b}{J}\right)s + \frac{Rb + K_b K_I}{LJ} = 0$$

- Multiplying by the prior equation by LJ we get:

$$LJs^2 + (JR + Lb)s + (K_I K_b + Rb) = 0$$

  – Note this is exactly the same characteristic equation we saw in the transfer function

  – Now use quadratic formula to solve for the two eigenvalues and use our 2nd order tools

Eigenvalues Analysis

- Most motors are heavily overdamped
  - This means their response can be approximated as first order
    - Due to the inductance being negligible (in terms of dynamics, i.e. eigenvalues)
  - Results in the following approximate characteristic equation:

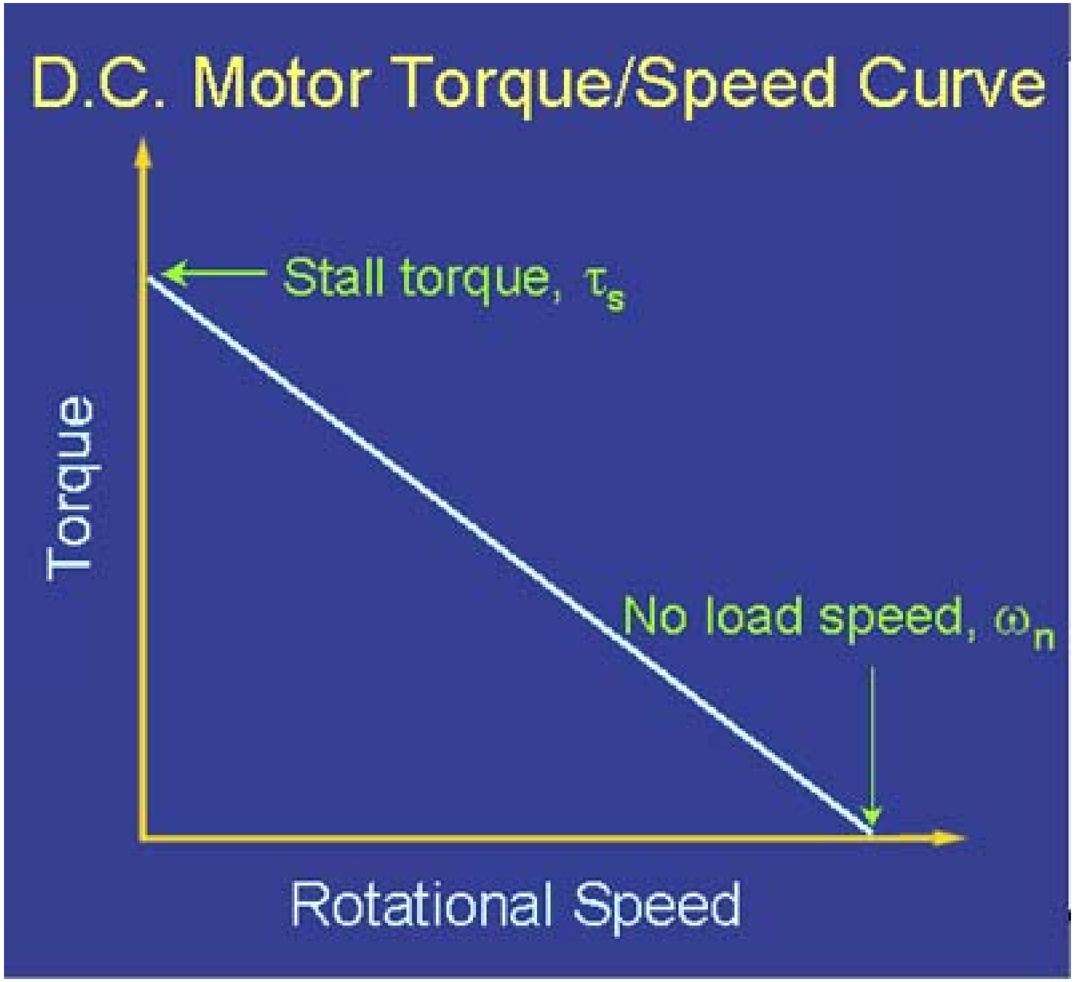
$$Js + (b + \frac{K_b K_I}{R}) = 0$$

  - Or the following differential equation:

$$J\dot{\omega} + (b + \frac{K_b K_I}{R})\omega = \frac{K_I}{R} V_s$$

Steady State Motor Speed Curve

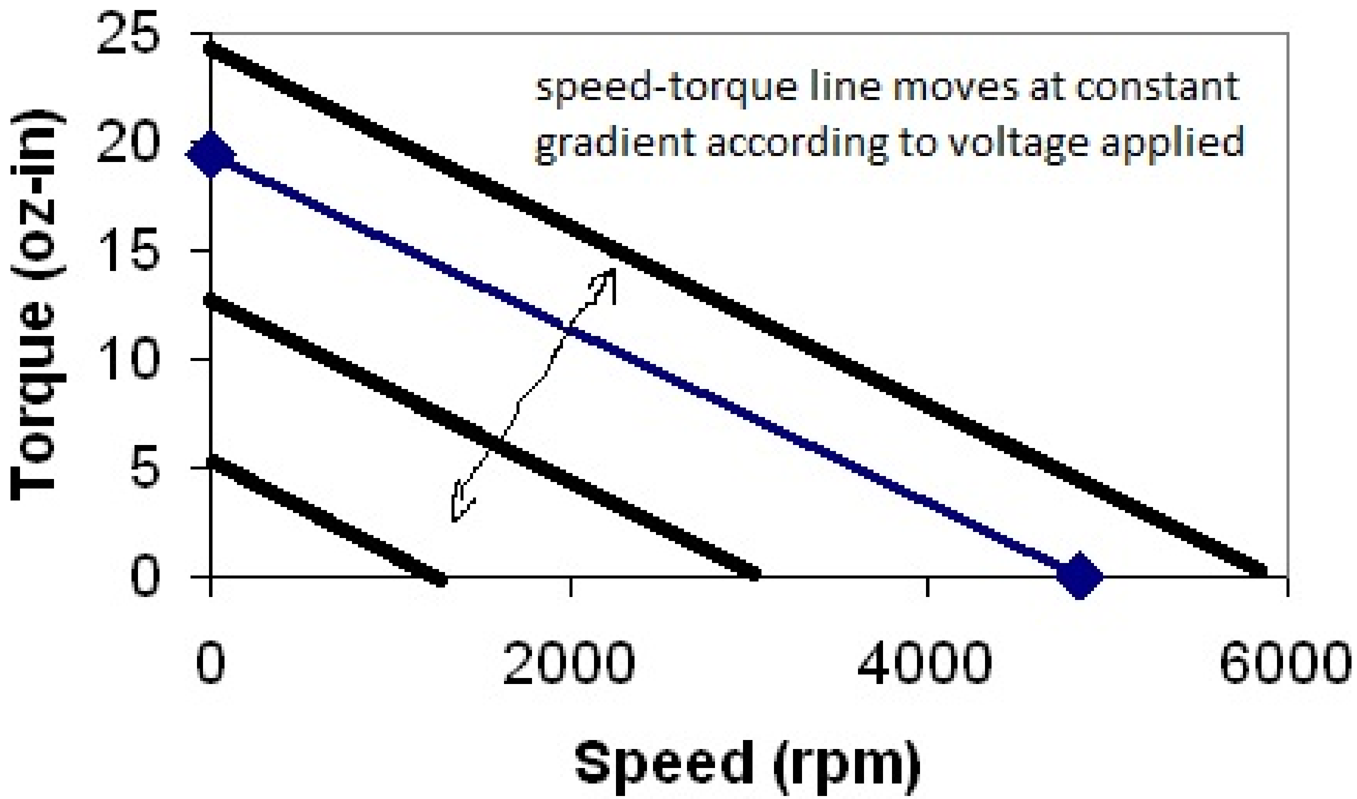
- DC Electric motors have a linear torque speed relationship at steady state
  - i.e., for $\dot{I} = \dot{\omega} = 0$
- You get max torque at zero speed
  - Called stall torque
  - Different from an IC engine
- You can only generate the max speed when there is no load
  - Called no load speed
  - The net torque at this speed is zero!



http://lancet.mit.edu/motors/motors3.html/

Motor Speed Curve

- Speed-Torque is shifted as more voltage is applied
  - Optional lecture will discuss how to vary the voltage



https://robotics.stackexchange.com/questions/4492/dc-motor-control-speed-torque-curve

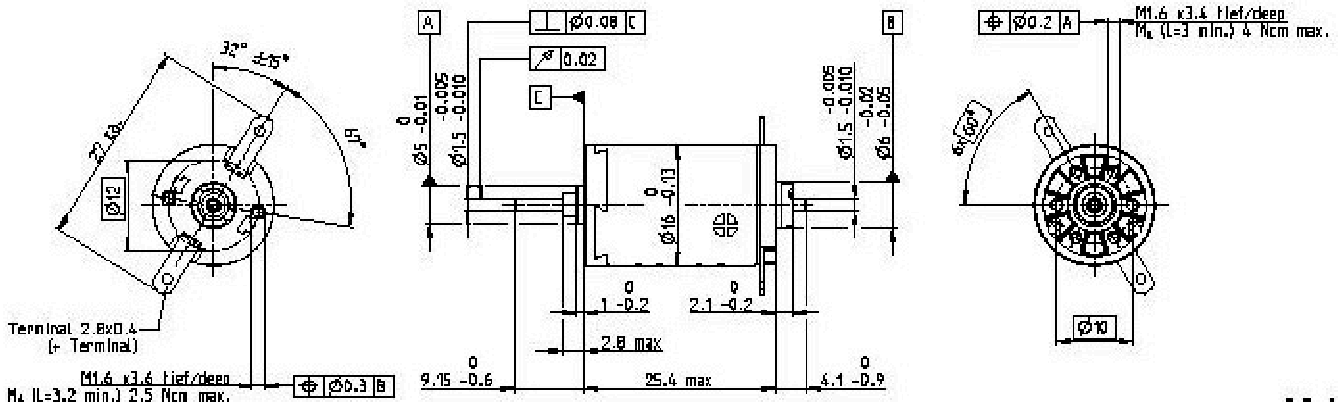
Maxon DC16 Spec Sheet

maxon DC-max

DC-max 16 S Precious Metal Brushes

DC motor ∅16 mm

Key Data: 2/4.3 W, 4.1 mNm, 11 000 rpm



M 1:1

Motor Data					
1	Nominal voltage	V	6	12	24
2	No load speed	rpm	7890	7560	7470
3	No load current	mA	14.7	6.90	3.40
4	Nominal speed	rpm	4830	4390	4210
5	Nominal torque (max. continuous torque)	mNm	4.06	3.92	3.80
6	Nominal current (max. continuous current)	A	0.577	0.267	0.128
7	Stall torque	mNm	10.5	9.44	8.75
8	Stall current	A	1.46	0.629	0.289
9	Max. efficiency	%	81	80	80
10	Terminal resistance	Ω	4.10	19.1	83.2
11	Terminal inductance	mH	0.140	0.610	2.49
12	Torque constant	mNm/A	7.19	15.0	30.3
13	Speed constant	rpm/V	1330	637	315
14	Speed/torque gradient	rpm/mNm	758	809	864
15	Mechanical time constant	ms	8.87	8.92	9.00
16	Rotor inertia	gcm²	1.12	1.05	0.994

https://www.maxongroup.com/medias/sys_master/root/8833376059422/19-EN-94.pdf