Calculating Gain and Phase from Transfer Functions

MECH 3140
Lecture #

Transfer Functions

- A differential equation $f(x, \dot{x}, \ddot{x}, \ldots) = u(t)$, has $u(t)$ as the input to the system with the output x
- Recall that transfer functions are simply the Laplace Transform representation of a differential equation from input to output:

$$H(s) = \frac{X(s)}{u(s)}$$

- Therefore it can be used to find the Gain and Phase between the input and output

Gain and Phase

- The gain and phase are found by calculating the gain and angle of the transfer function evaluates at jω

$$G(\omega) = |H(j\omega)| = \frac{|num(j\omega)|}{|den(j\omega)|}$$

$$\emptyset(\omega) = \angle H(j\omega) = \angle num(j\omega) - \angle den(j\omega)$$

Example #1 (solving the Diff Eq)

Fall 2019 Exam #1, Bonus Problem:

$$\dot{x} + x = \dot{f} + f \qquad\qquad f(\omega) = \sin(\omega t)$$

Recall we can represent a sinusoid in the following format:

$$f = Im\{1e^{j\omega t}\}$$
$$x = Im\{He^{j\omega t}\}$$

Then taking the derivatives:

$$\dot{x} = Im\{Hj\omega e^{j\omega t}\}$$
$$\dot{f} = Im\{1j\omega e^{j\omega t}\}$$

$$Im\{Hje^{j\omega t} + He^{j\omega t} = 1e^{j\omega t} + 1j\omega e^{j\omega t}\}$$

Example #1 (solving the Diff Eq)

Solving for H (which is the output):

$$H = \frac{j\omega + 1}{j\omega + 1}$$

$$G = \left|\frac{H}{U}\right| = \frac{\sqrt{\omega^2 + 1^2}}{\sqrt{\omega^2 + 1^2}} = 1$$

Example #1 (using Transfer Function)

Fall 2019 Exam #1, Bonus Problem:

$$\dot{x} + x = \dot{f} + f \qquad\qquad f(\omega) = \sin(\omega t)$$

Take the Laplace of the entire equation and setting initial conditions to zero (since we are solving for the transfer function):

$$sX(s) + X(s) = sF(s) + F(s)$$

$$X(s)(s+1) = F(s)(s+1)$$

$$\frac{X(s)}{F(s)} = \frac{s+1}{s+1}$$

Example #1 (using Transfer Function)

Now, to find the gain simply evaluate the transfer function at $j\omega$

$$\frac{X(s)}{F(s)} = \frac{s+1}{s+1}$$

$$G = \left|\frac{j\omega + 1}{j\omega + 1}\right| = \frac{\sqrt{\omega^2 + 1^2}}{\sqrt{\omega^2 + 1^2}} = 1$$

Example #2 (solving the Diff Eq)

Spring 2020 Exam #1, Bonus Problem:

$$\ddot{x} + 25x = u(\mathrm{t})$$

Recall we can represent a sinusoid in the following format:

$$u = Im\{Ue^{j\omega t}\}$$
$$x = Im\{He^{j\omega t}\}$$

Then taking the derivatives:

$$\dot{x} = Im\{Hj\omega e^{j\omega t}\}$$
$$\ddot{x} = Im\left\{H(j\omega)^2 e^{j\omega t}\right\} = Im\{-H\omega^2 e^{j\omega t}\}$$

$$Im\{-H\omega^2 e^{j\omega t} + 25He^{j\omega t} = Ue^{j\omega t}\}$$

Example #2 (solving the Diff Eq)

Solving for H (which is the output):

$$H = \frac{U}{-\omega^2 + 25}$$

$$G = \left|\frac{H}{U}\right| = \frac{1}{25 - \omega^2}$$

Example #2 (using Transfer Function)

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Spring 2020 Exam #1, Bonus Problem:

$$\ddot{x} + 25x = u(\mathrm{t})$$

Take the Laplace of the entire equation and setting initial conditions to zero (since we are solving for the transfer function):

$$s^2X(s) + 25X(s) = U(s)$$

$$X(s)(s^2 + 25) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^2 + 25}$$

Example #2 (using Transfer Function)

Now, to find the gain simply evaluate the transfer function at jω

$$\frac{X(s)}{U(s)} = \frac{1}{s^2 + 25}$$

$$G = \left| \frac{1}{(j\omega)^2 + 25} \right| = \frac{1}{25 - \omega^2}$$

Calculating Gain and Phase in Matlab

- Matlab uses transfer functions to calculate gain and phase and generate bode plots
- Recall that there are 2 ways to plot data logarithmically
  - 1) Plot on a log scale
  - 2) Take the log of the data & plot on normal scale
  - Matlab does both (just to be annoying or to ensure you can do both – actually its just the standard of how bode plots are shown).

Calculating Gain and Phase in Matlab

- Matlab plots the Gain in decibels (db):

$$1\ db = 20log_{10}(G)$$

- Note the following about db:

G=0 db (Gain=1, output=input)

G>0 db (Gain>1, output>input)

G<0 db (Gain<1, output<input

Calculating Gain and Phase in Matlab

- Matlab code:

```
>> num=1;
>> den=[1 0 25];
>> sys=tf(num,den)
>>bode(sys)
```

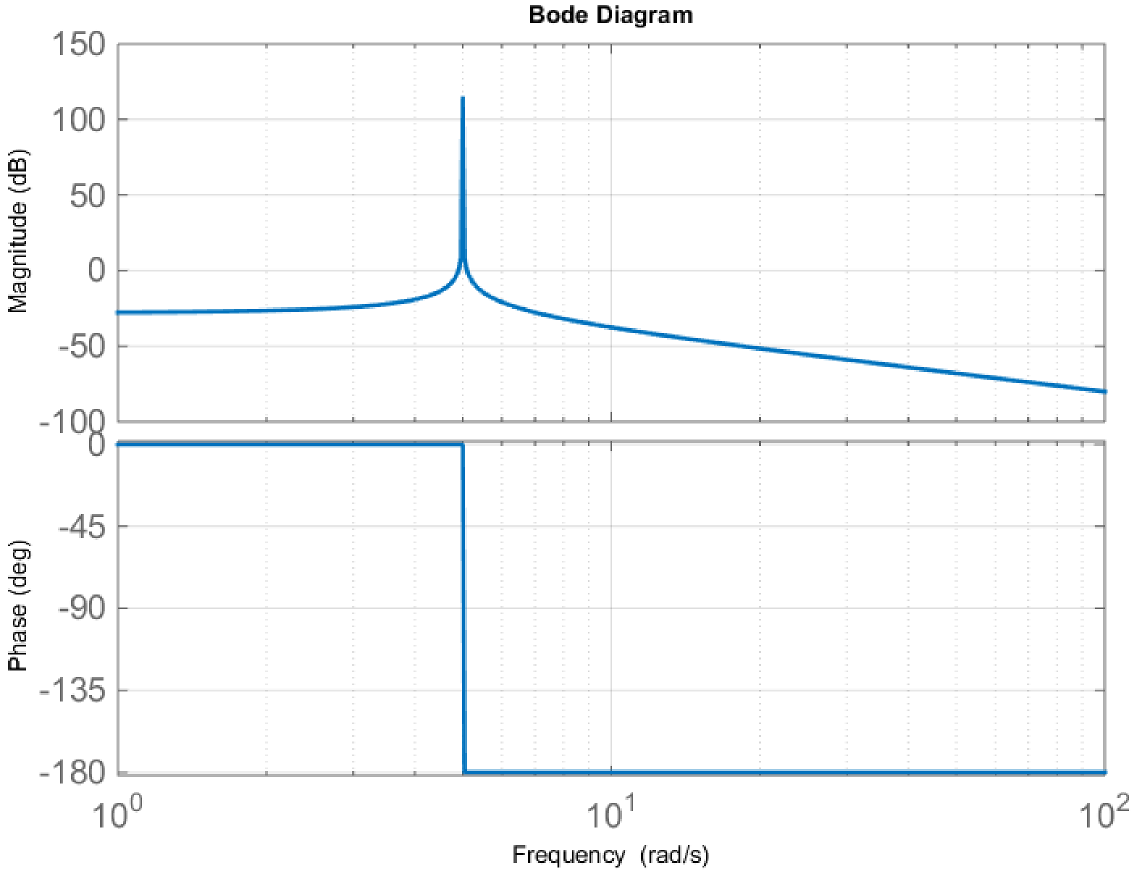
% $s^2 + 0s + 25$

- If you want the actual gain you can do:

```
>>[gain, phase, freq]=bode(sys)
```

Calculating Gain and Phase in Matlab

- Note the spike at ω=5 rad/s



Example #3

- PI Cruise Control from a few lectures ago:

$$m\dot{V} + bV = F$$

$$K_p e + K_I \int e dt = F$$

  - Taking the laplace of each equation (with IC=0 and e=r-v):

$$msV(s) + bV(s) = F(s)$$

$$Kp\big(R(s) - V(s)\big) + \frac{K_I}{s}(R(s) - V(s)) = F(s)$$

Example #3

– Substituting in F(s) and collecting terms

$$V(s)\left(ms + b + K_p + \frac{K_I}{s}\right) = R(s)\left(K_p + \frac{K_I}{s}\right)$$

– Then solving for the transfer function:

$$\frac{V(s)}{R(s)} = \frac{K_p + \frac{K_I}{s}}{\left(ms + b + K_p + \frac{K_I}{s}\right)}$$

$$= \frac{K_p s + K_I}{ms^2 + (b + K_p)s + K_I}$$

Example #3

- This is called the closed loop transfer function
  - It is from the reference input to the velocity output
  - Notice the DC Gain is one (which means for a constant reference, the steady state velocity will equal the reference
  - Notice the PI controller adds a "zero" (root in the numerator) and a "pole"
    - So the total order is 2nd order (2 poles or 2nd order denominator)

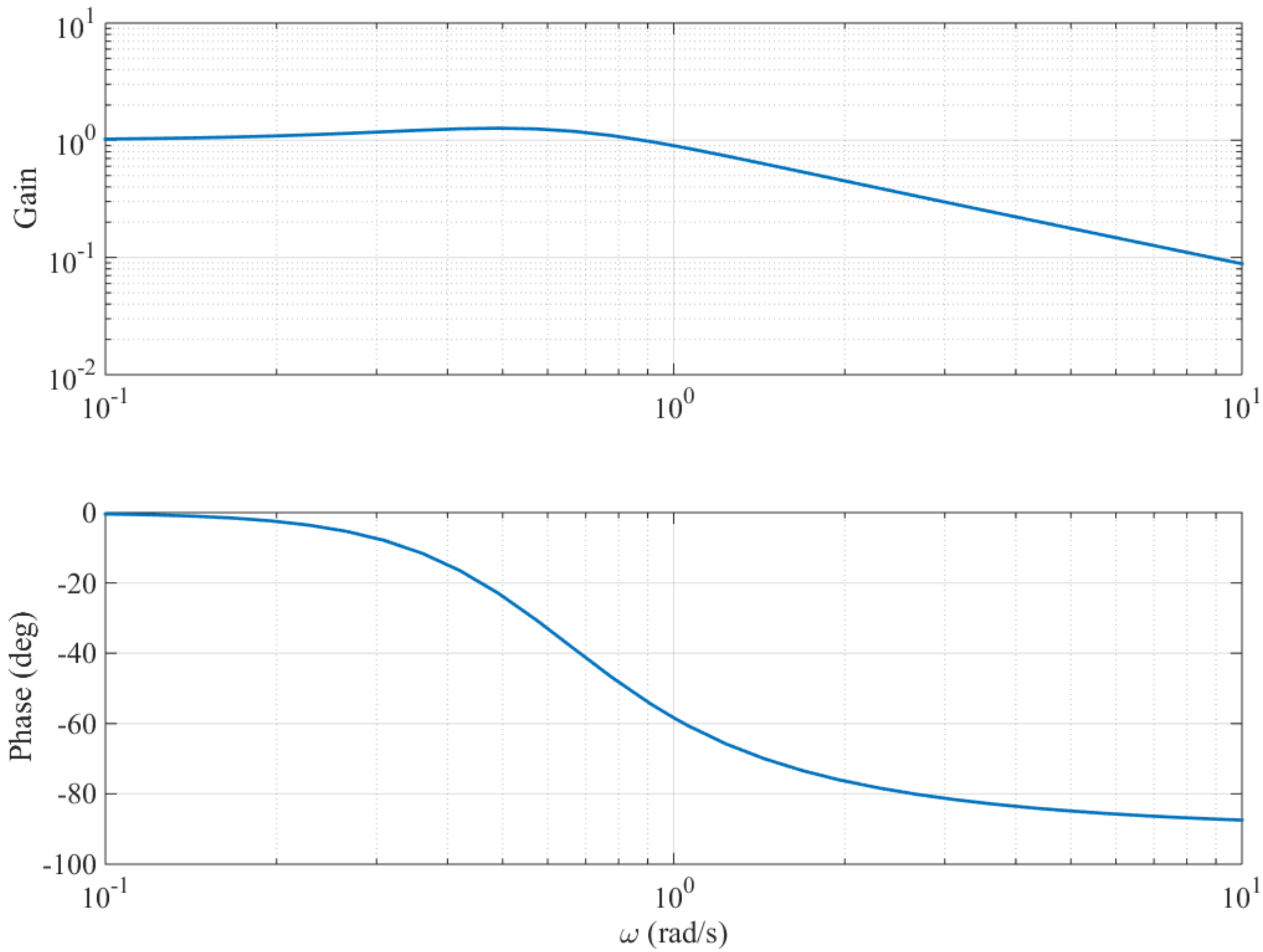
Example #3

Now to find the gain and phase:

- $G(\omega) = \left|\frac{K_p j\omega + K_I}{m(j\omega)^2 + (K_p + b)(j\omega) + K_I}\right| = \frac{\sqrt{(K_p\omega)^2 + (K_I)^2}}{\sqrt{(K_p\omega + b\omega)^2 + (K_I - m\omega^2)^2}}$

- $\emptyset(\omega) = tan^{-1}\left(\frac{K_p\omega}{K_I}\right) - tan^{-1}\left(\frac{K_p\omega + b\omega}{K_I - m\omega^2}\right)$

Example #3 (Bode Plot)



m=1800 kg
(me in a mustang)
b=10 Ns/m
(made up)

Kp=1,589 Ns/m
Ki=710 N/m

Example #3

Lets assume R=10sin(1t).

What would the car's velocity be?

- $G(1) = \frac{\sqrt{(1589)^2+(710)^2}}{\sqrt{(1599)^2+(710-1800)^2}} = 0.899$
- $\emptyset(1) = tan^{-1}\left(\frac{1589}{710}\right) - tan^{-1}\left(\frac{1599}{710-1800}\right) = -58\ deg$

$$V(t) = 8.99\sin(1t - 1.019)$$