Review of Differential Equations

Solving Differential Equations
Analytically & Numerically with Matlab
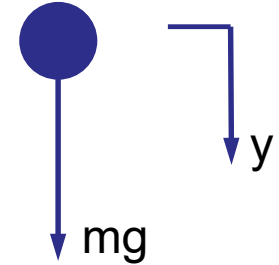
Start with a Simple Example

- Ball falling in a vacuum
- We can use Newton's 2nd Law to obtain the equation of motion for the ball

$$\sum F_y = ma = m\ddot{y}$$

$$mg = m\ddot{y}$$
$$\ddot{y} = g$$

- But now what? What does this tell us?
  - As engineers we want to know what happens to the ball over time
    - Its Velocity
    - Its Position
    - Maybe its velocity at a certain position (height)
  - So how do we obtain these things from the "model" $mg = m\ddot{y}$?
    - We can use calculus to integrate the acceleration with respect to time!

Solving for Velocity

- So integrating the acceleration with respect to time

$$v(t) = \int g\,dt$$

- Since gravity is constant over time:

$$v(t) = \int g\,dt = gt + A$$

- What is A?
  - Its an initial condition (IC) or boundary condition coefficient
    - Need to know the velocity at some point to solve for A
    - Notice: 1 integration, 1 IC

Solving for Position

- Integrate the velocity with respect to time to get position

$$v(t) = \int (gt + A)dt = \frac{1}{2}gt^2 + At + B$$

- What is B?
  - Its another initial condition (IC) or boundary condition coefficient
    - Need to know the position at some point to solve for B
    - Notice: 2 integrations, 2 ICs (A & B)
- Of course this is a well known equation that many of you have unfortunately memorized $(\frac{1}{2}at^2 + v_0t + x_0)$

  - Unfortunate because it is so easy to derive
  - Unfortunate because it is doubtful that by memorizing the final answer you know the assumptions
    - Constant acceleration with no other forces (i.e. a ball in a vacuum).

Lets add some more reality

- I don't know about you, but I none of the systems I work on operate in a vacuum!

- So lets add some realistic loss (such as air drag)
  - If you have had fluid mechanics, you know that air drag is proportional to velocity **squared**
  - However, for this example I am going to use a simplified loss that is simply proportional to velocity
    - Still much more realistic than a vacuum!
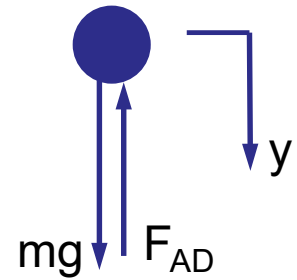
$$F_{AD} = bv$$

- Again using Newton's 2nd law on our FBD:

$$\sum F_y = ma = m\ddot{y}$$

$$mg - F_{AD} = m\ddot{y}$$
$$mg - bv = m\ddot{y}$$

Lets add some more reality

- Now we have a model of the ball with some resistance force acting on it.
  - But again, what we will want things like velocity and position of the ball over time.
- So how do we get this?
  - We can't just integrate like we did last time? Why not?
    - Well if we try and integrate acceleration to get velocity we have:

$$\int \ddot{y} = \int g dt + \int \frac{b}{m} v dt$$

  - But what is the integral of velocity with respect to time ($\int v dt$)
    - Its not vt because velocity is not constant!
    - If you say its position (which is true), you have not solved for velocity as a function of time because now your velocity result is a function of position (which we don't know)!   And if you say we will integrate velocity to get position then that will be a function of the integral of positon!
    - So you can't use Calculus! This is why we make you take an ENTIRE class to solve this type of problem
      - *ITS CALLED DIFFERENTIAL EQUATIONS!!!*

Looking at our new problem

- Lets rewrite our equation

$$mg - bv = m\ddot{y}$$

- Into the form:

$$m\ddot{y} + bv = mg$$

- Then rewriting velocity we get:

$$m\ddot{y} + b\dot{y} = mg$$

- And now we see that our "model" turns out was a differential equation.
  - This is why engineers take differential equations – it turns out that when you model engineering systems (fluids, thermodynamics, heat transfer, dynamics), the governing equations that describe what happens are differential equations.

Looking at our differential equation

$$m\ddot{y} + b\dot{y} = mg$$

- Some questions about our differential equation:
    - What is its order?
        - It looks to be 2nd order. But in fact this differential equation is fundamentally 1st order. This is because I can re-write this one in terms of velocity (you can't always do this):

        $$m\dot{v} + bv = mg$$

        - Some might says its 2nd order in position or 1st order in velocity. But the fundamental differential equation that must be solved is 1st order (velocity). From there I can solve for position by just integrating the solved velocity.
    - Is it linear or non-linear?
        - Its linear. This is why I didn't use real air drag (that would have made the differential equation non-linear):

        $$m\dot{v} + bv^2 = mg$$

    - Is it homogeneous or non-homogeneous (i.e. does it have an input or is it being forced or is it just "free")?
        - There is an input (mg), therefore the differential equation is non-homogeneous.
    - What is its output (i.e. what would be solving for)?
        - The fundamental differential equation's output is velocity.
        - The original differential equation's output is position.
    - Is it Time Invariant (i.e. does the differential equation change over time)?
        - If m, b, and g are constant (which for our ball they should be), then the differential equation is time invariant.

Solving our differential equation

$$mv̇ + bv = mg$$

- Many ways to solve differential equations analytically (i.e. with an exact closed-form solution)

  - Separation of variables (write $\dot{v} = \frac{dv}{dt}$)
    - Applicable for 1st order differential equations
    - Can be used to solve our problem with the non-linear air drag term!

  - Undetermined Coefficients (i.e. "guessing" a solution)
    - The method I prefer

  - Laplace Transforms
    - Converts differential equations from the time domain to the frequency domain
    - Allows use of algebra to manipulate the equations
    - Generally requires use of Laplace tables to solve the differential equations
    - We will actually use Laplace Transforms later for representing differential equations

Guess Solution

$$m\dot{v} + bv = mg$$

- First we will "guess" a solution to the homogenous part of the differential equation
  - Known as the homogenous solution or the unforced or free solution
  - Set the input to zero:
  $$m\dot{v} + bv = 0$$

- We could guess many solutions and check to see if they satisfy the above equation (remember the solution is velocity as a function of time that satisfies the governing equation)
  - Lets try v(t)=10
  $$m(0) + 10b \neq 0$$
  - Now v(t)=0 will work (but that is call the "trivial" solution – i.e. nothing)
  $$m(0) + 0b = 0$$

Guess Solution

$$m\dot{v} + bv = 0$$

- Lets get smarter with our guesses. We need something that its derivative can cancel with itself so that $m\dot{v}$ can cancel bv

- In other words the solution has to be a form that when we take the derivative we get the same thing back.
  - So we will guess a solution of the form: $v(t) = Ae^{st}$
  $$\dot{v}(t) = Ase^{st}$$

- Now lets plug into our differential equation and see if it works as a solution:
  $$m(Ase^{st}) + bAe^{st} = 0$$
  $$Ae^{st}(ms + b) = 0$$

- So this does work (i.e. the solution $v(t) = Ae^{st}$ satisfies the differential equation $m\dot{v} + bv = 0$) if:

$$ms + b = 0 \quad \text{(i.e. s=}-\frac{b}{m}\text{)} \qquad \text{OR} \qquad A = 0$$

Characteristic Equation & Eigenvalues

- Turns out that the equation $ms + b = 0$ is so important it gets its own name
  - Characteristic Equation
  - Its solution provides the characteristics of the response of the differential equation
- And the solution to the characteristic equation is so important it gets its own name
  - Eigenvalue(s)!
  - You have one non-zero eigenvalue per fundamental order of the differential equation
  - MECH3140 is all about eigenvalues!

Particular Solution

- So we now have the homogenous solution:

$$v_h(t) = Ae^{-\frac{b}{m}t}$$

- Now we have to solve for the particular solution or forced solution of the differential equation

- For this we will guess a solution that is the form of the input plus all of its derivatives:

Input	Particular Solution Form
c	a
ct	$a_1 t + a_2$
ct^2	$a_1 t^2 + a_2 t + a_3$
$c\sin(\omega t)$	$a_1 \sin(\omega t) + a_2 \cos(\omega t)$

Common Types of inputs

- Impulse
  - Modeled by the dirac delta function (δ)
    - The dirac delta function is infinitely tall and infinitesimally narrow, such that the area under it is equal to one

    $$\int_{-\infty}^{\infty} \delta dt = 1$$

  - True impulses do not exist
  - Usually modeled by a chance in initial condition after the impulse (using conservation of energy)

- Constant
  - True constants rarely exist or are not of interest to us in dynamics
    - If the input has been on for all time, then the system would be in steady state (equilibrium) and all the interesting transients are done
   - Even our ball example does not have a true constant input

- Step Input
  - This is a constant after some start time (usually t≥0)
  - The equivalent of turning on the input (i.e. flipping a switch)
    - For our ball example, this would be letting go of the ball. Prior to dropping the ball someone was holding it with a force equation to weight, meaning the **total** input was zero. Once dropped, **then** the input became "mg"

$$u(t) = \delta(t)$$

$$u(t) = c$$

$$u(t) = \begin{cases} 0 & t < 0 \\ c & t \geq 0 \end{cases}$$
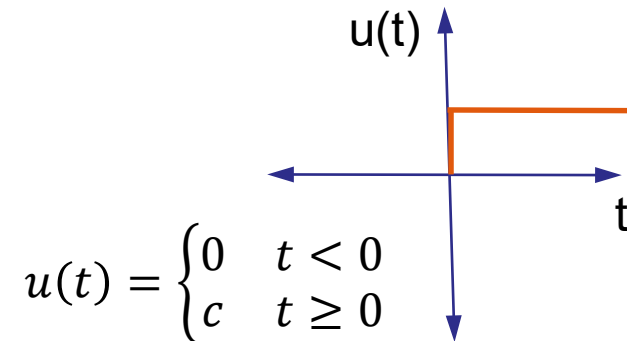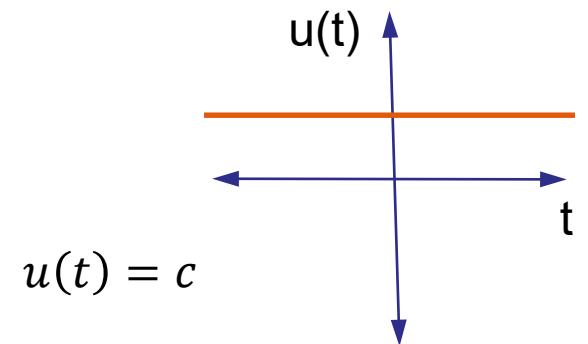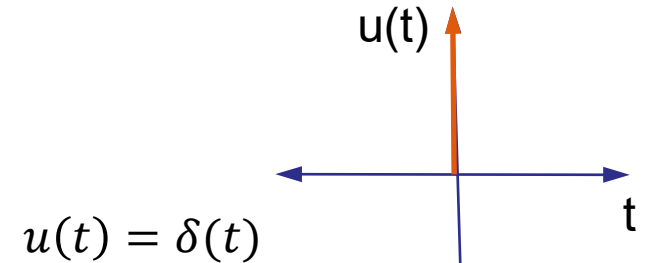
Common Types of inputs

- Ramp Input
  - Grows linearly over time after some start time (usually t>0)

$$u(t) = \begin{cases} 0 & t < 0 \\ ct & t \geq 0 \end{cases}$$

- Sinusoidal Input
  - You need to be able to measure and understand frequency!
  - You need to be able to go between Hz (cycles/sec) and rad/sec

$$u(t) = A sin(\omega t)$$

- Note that inputs can also be combinations of multiple inputs (i.e. a sinusoid plus a constant or a step plus a ramp)
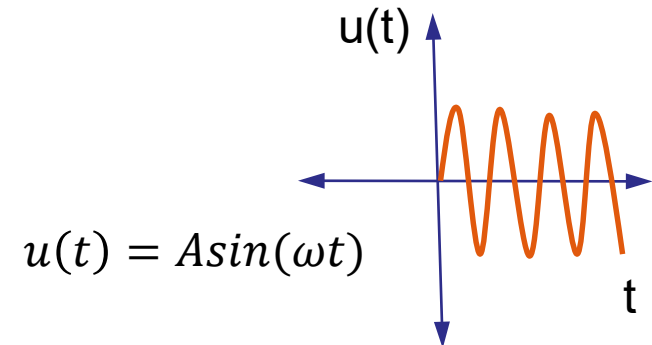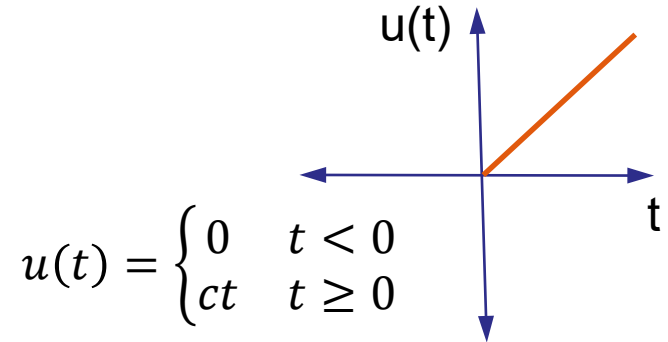
Particular Solution

$$m\dot{v} + bv = mg$$

- Our differential equation has a constant input (mg)
  - At least from the time the ball was released (i.e from t=0)
- So we will guess a constant for the particular solution:
$$v_p(t) = c$$

- Plugging that solution into our differential equation we get:
$$m(0) + bc = mg$$

- Therefore:

$$v_p(t) = c = \frac{mg}{b}$$

Total Solution

$$mv̇ + bv = mg$$

- The total solution is simply the sum of the particular and homogeneous solutions:

$$v(t) = Ae^{-\frac{b}{m}t} + \frac{mg}{b}$$

- But wait, we have an unknown coefficient (A)!

- Any ideas how we solve for that?
  - Its our boundary condition or initial condition coefficient
  - Exactly like when we integrate!
  - Note we will get one IC coefficient for every order of the differential equation we solved!

Solving for the IC Coefficient

$$m\dot{v} + bv = mg$$

- Note that you can not solve for the IC coefficient until you have solved for BOTH the homogeneous and particular solutions

$$v(t) = Ae^{-\frac{b}{m}t} + \frac{mg}{b}$$

- Lets assume the ball was let go from rest:

$$v(0) = 0$$

- Then

$$0 = Ae^0 + \frac{mg}{b}$$

- Therefore:

$$A = -\frac{mg}{b}$$

- And now we have our final solution:

$$v(t) = \frac{mg}{b} - \frac{mg}{b}e^{-\frac{b}{m}t}$$

Solution

- So we started with the following governing equation:
$$m\dot{v} + bv = mg$$

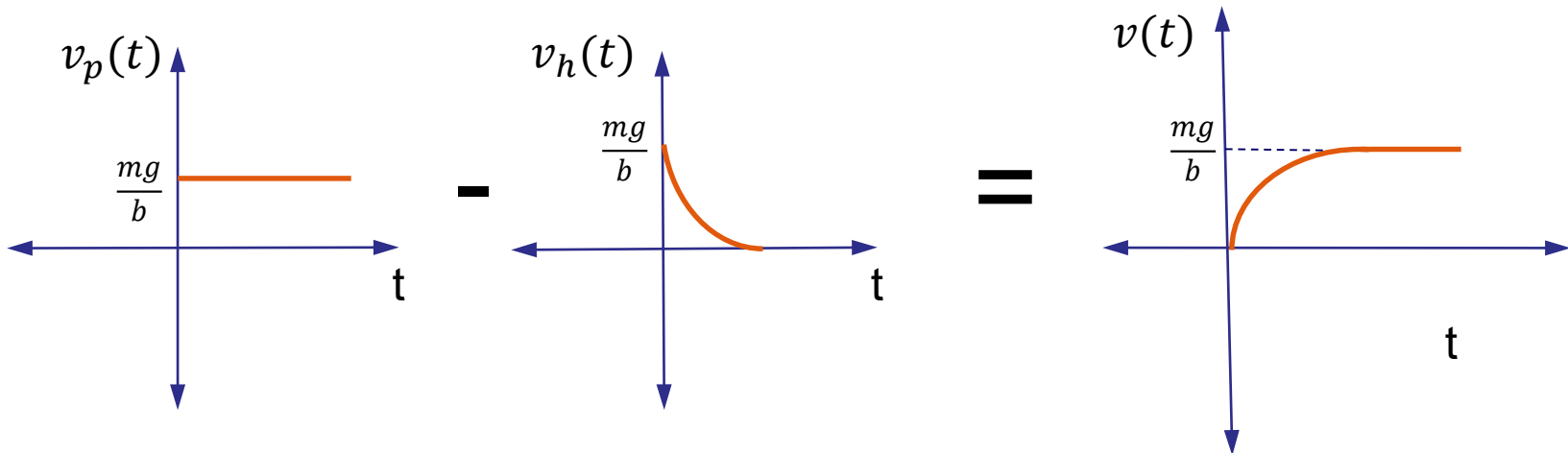- And were able to find the solution to the velocity of the ball:
$$v(t) = \frac{mg}{b} - \frac{mg}{b}e^{-\frac{b}{m}t}$$

- Having now solved the fundamental differential equation, we could now integrate (or differentiate) the solution with respect to time to get the position (or acceleration) of the ball as a function of time.

$$v(t) = \frac{mg}{b} - \frac{mg}{b} e^{-\frac{b}{m}t}$$

- Lets see what the solution looks like over time

A different system

$$m\ddot{x} + kx = 0$$

- Consider the system above
  - What is the order?
    - 2nd Order
  - Homogeneous or non-homogeneous
    - Homogeneous (there is no input)
  - Linear or non-linear
    - Linear
  - What is the output?
    - x (which in this class is usually position)
- What can we use for the homogeneous solution?
  - $x(t) = Ae^{st}$
    - This will work, but ….

A different system

$$m\ddot{x} + kx = 0$$

- We can also see that we need to the 2nd derivative to cancel with itself.

- Therefore, there is another function that we can use in this case for the homogeneous solution
  - $x(t) = A\sin(\omega t)$
    - Note since an exponential and sinusoid will both satisfy the above differential equation, a sinusoid must be representable by an exponential (known as Euler's identity – look it up if you have forgotten it).

- Taking derivatives of our "guess" solution to plug in:
$$\dot{x}(t) = A\omega\cos(\omega t)$$
$$\ddot{x}(t) = -A\omega^2 \sin(\omega t)$$

A different system

$$m(-A\omega^2 sin(\omega t)) + kA sin(\omega t) = 0$$
$$A sin(\omega t)(m\omega^2 + k) = 0$$
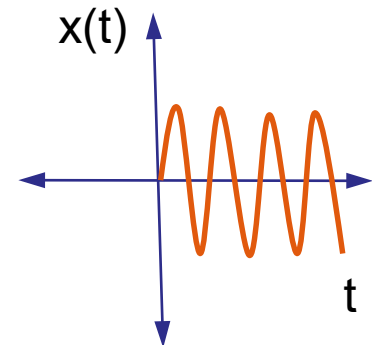
- Therefore, the sinusoid fits the solution if:

$$(m\omega^2 + k) = 0$$

or

$$\omega = \sqrt{\frac{k}{m}}$$

- So our solution is:

$$x(t) = A sin\left(\sqrt{\frac{k}{m}} t\right)$$

x(t)

t

A different system

- Think about how the solution to the two different differential equations differs (that is a lot of "diff" words in one bullet).
  - The first problem had a homogeneous solution that was an exponential decay
  - The second problem had a homogeneous solution that was oscillatory (i.e. a sinusoid).
- By solving these two different differential equations, we now have some incredible insight (if we pay attention to our answers) that we can use going forward without having to solve the differential equation!
  - We can start to understand what kind of model would be needed (or what the differential equation is) for a system we observe by knowing this difference
  - We can understand what the response will be for a system that we have modeled before ever solving the differential equation
  - We can figure out how to BUILD a system to respond/behave in a certain way by knowing this difference in solutions to differential equations
- Pretty cool (and powerful) when you think about it!

Numerical Solutions

- Another way to ***approximate the solution*** to a differential equations is to use numerical methods

- In this method, we solve for the highest derivative, assume it is constant over some small period of time (which is not true since the derivative is actually a function of the integral) and take the area under the curve to integrate that derivative.

- So for our problem we would re-write it as:

$$\dot{v} = \frac{b}{m} v + \frac{mg}{m}$$

  - Given an initial velocity, the acceleration (i.e. $\dot{v}$) can be calculated
    - Note that when solving the differential equation you still need initial conditions!
  - This acceleration can then be numerically integrated using one of several methods:
    - Euler's Integration
    - Trapezoidal
    - Runge-Kutta (i.e. ode23 and ode 45)
    - Simpson

Numerical Solutions

- An illustration of the numerical integration is shown below
  - Note that the acceleration is not known for all time since it is a function of velocity (i.e. it's a function of the area under the curve)!
  - Since the velocity is the area under the acceleration, the "new" velocity can be found over time by summing up the area under the curve
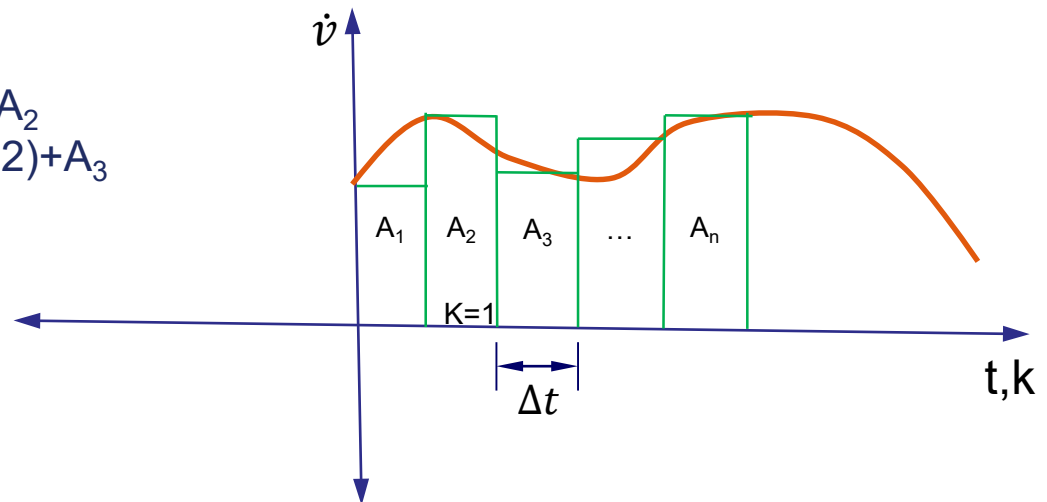
$$v_k = \sum_{i=1}^{k} A_i$$

  - Using simple Euler Integration of height ($\dot{v}$) times width (Δt) each area (and thus velocity at each time step) can be calculated from: $A_i = \dot{v}_i \Delta t = \left(\frac{b}{m} v_i + \frac{mg}{m}\right) \Delta t$
 - The velocity at each time step is simply the sum of the areas:

v(k=1)=A_1+v(0)
v(k=2)=A_1+A_2+v(0)=v(k=1)+A_2
v(k=3)=A_1+A_2+A_3+v(0)=v(k=2)+A_3

Numerical Solutions

- Several errors that occur in numerical solutions
  - The "missed" area under the curve (as seen in the diagram)
    - This can be corrected by using a smaller sample rate (i.e. smaller Δt) or higher order polynomial than just rectangles (i.e. trapezoidal, Runge-Kutta, etc.)
  - The assumption that $\dot{v}$ is constant over the period Δt
    - $\dot{v}$ is a function of velocity (which is the area under the curve) and therefore can not be constant!
    - This error is also minimized with small sample rate

- Note that using small sample rates increase the number of times you must calculate the area over a given time period
  - This increased the length of time to run the numerical integration
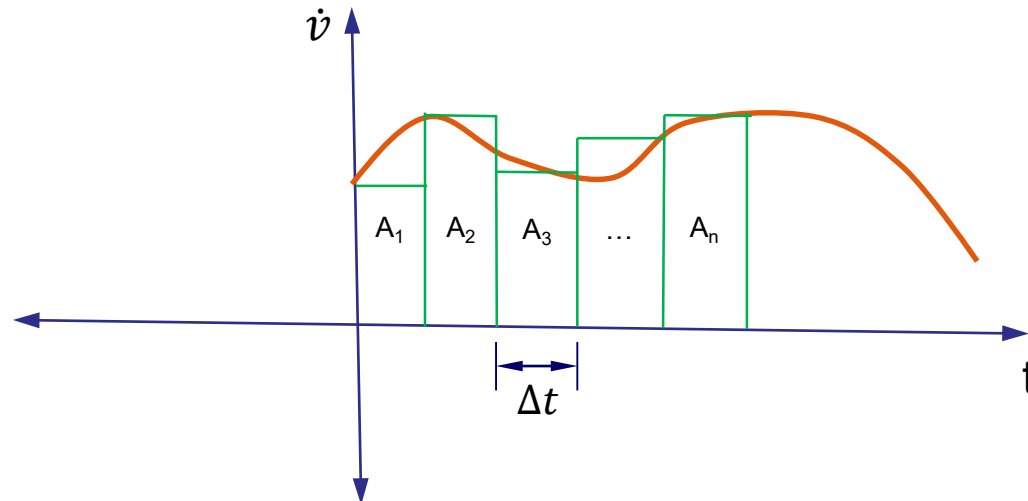
Matlab Code

```
clear all;close all;
% Initialize all variables
m_ball=5;              % mass in kg
b_ball=3;              %resistance in Ns/m
gc=9.81;               %gravitation constant in SI
end_time=10;           %end time in seconds
del_t=1/1000;          %integration sample rate in seconds

LN=end_time/del_t;    %number of samples to run the simulation

% Initialize matrix sizes since matlab does not allocate memory ahead of time
v_dot(1:LN)=NaN;
vel(1:LN)=NaN;
time(1:LN)=NaN;
%Initial Conditions
time(1)=0              %start time in sec
vel(1)=0               % initial velocity in m/s

for index=1:LN
    v_dot(index)=-(b_ball/m_ball)*vel(index)+(m_ball*gc/m_ball);          % calculating the accel
    vel(index+1)=vel(index)+v_dot(index)*del_t;        % calculating the area & adding to prev area to get vel
    time(index+1)=time(index)+del_t;
end

plot(time,vel); xlabel('Time (sec)');ylabel('Ball Velocity (m/s)');grid;
```

Some Matlab Tips

- ## Define all your variables first
  - Don't calculate values that matlab will calculate
  - Write your code in terms of all your variables so you can easily change them.
  - Example: Instead of m=10; g=9.81; weight=98.1;
                    Use: m=10; g=9.81; weight=m*g;

- ## Define the size of matrices upfont
  - Matlab allocates memory as it "needs" it.
  - This will decrease run time

- ## Avoid generic variable names like "k"
  - You would not believe the number of times students have used k for spring and k for the index and can't figure out why their code doesn't work when they had a spring that was changing from 1 to 100 (i.e. for k=1:100)

Solving with Laplace Transforms

- Laplace Transforms also provide a method to solve differential equations
  - This method usually requires look up tables and sometimes difficult partial fraction expansion to use the look up tables.