GPS Satellite Orbits

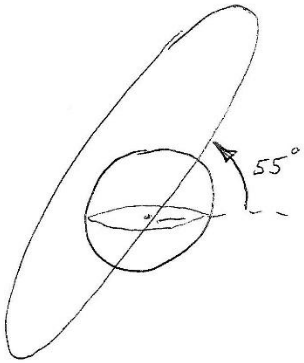
MECH 5970/6970
Fundamentals of GPS

Overview

- GPS consists of 24+ satellite vehicles (SVs)
- The orbits are:
  - 6 orbital planes
    - 55 degree inclination angles
      - less coverage at poles
  - Approximate circular orbits
  - 12 sidereal hour orbits
    - SV position repeats approximately every 23:56 hours
    - 20,162 km from equator
    - 26,561 km from center of earth
    - Travel at approximately 2.7 km/sec

GPS Satellite Overview

GPS Satellite Orbits

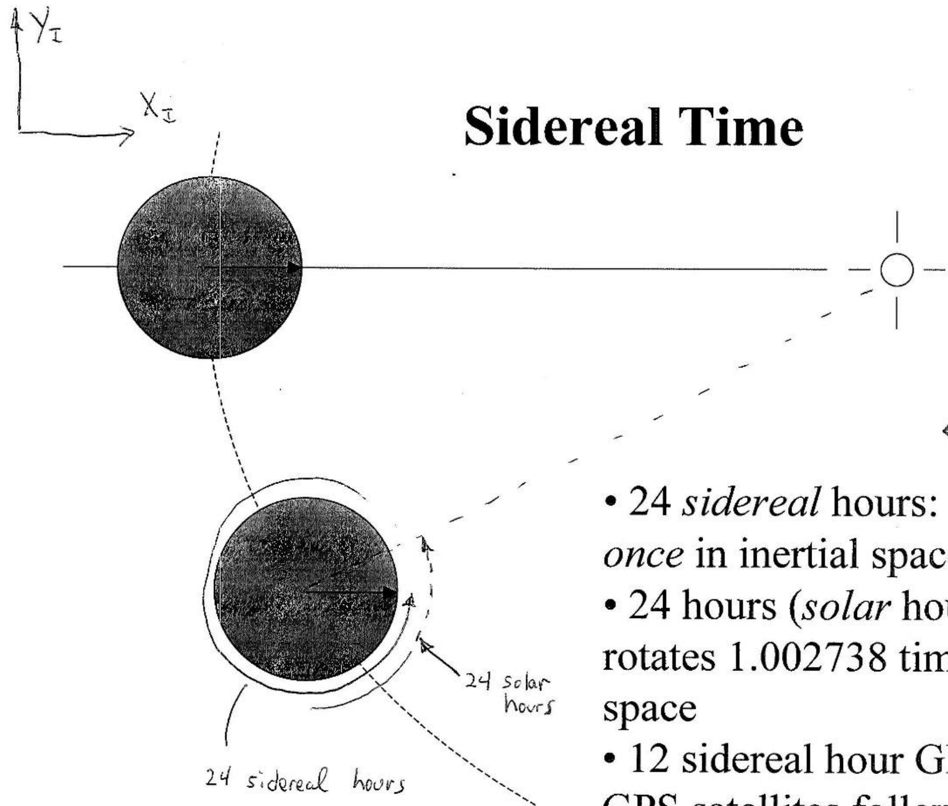


- 24+ satellites
  (space vehicles or SVs)
- 6 orbital planes
- 55 degree inclination
- (Mostly) circular orbits
- 12 sidereal hour orbits
- 26,561 km from earth's center
- 20,162 altitude (equatorial)
- 2.7 km/second

How do we figure out where the satellites in view are right now (i.e., how do we get a good estimate of X?)

GPS Sidereal Time

Sidereal Time

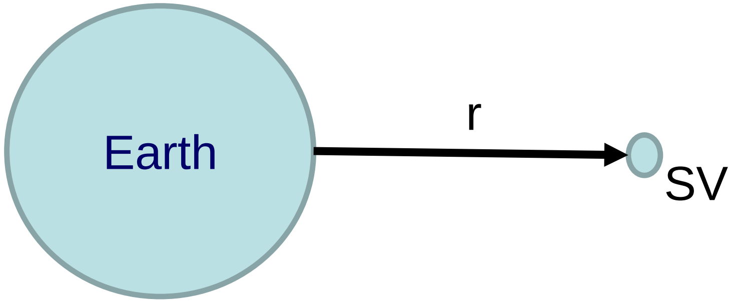


- 24 *sidereal* hours: Earth rotates *once* in inertial space
- 24 hours (*solar* hours): Earth rotates 1.002738 times in inertial space
- 12 sidereal hour GPS orbit --> GPS satellites follow (roughly) the same ground tracks every day!

Newton vs. Kepler

$$\mu_E = 3{,}986{,}004.418 \ \frac{m^3}{s^2}$$

$$G = 6.674 \times 10^{-11} \ \frac{m^3}{kgs^2}$$



$$F = \frac{GM_E m_{sv}}{r^2} = \frac{\mu_E m_{sv}}{r^2}$$

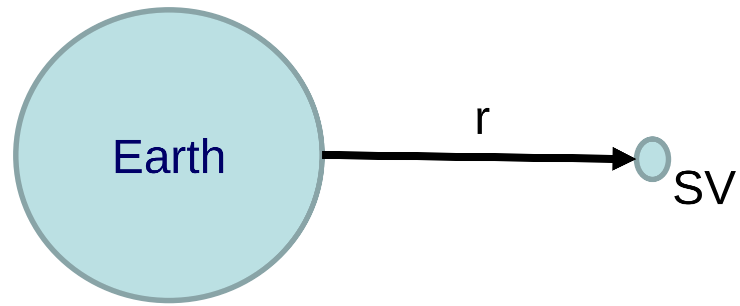
- Using Newton's Laws: $\Sigma F = m\ddot{x}$
  - SV:

$$m_{sv}\ddot{\vec{r}}_s = \frac{-GM_E m_{sv}}{r^2} \cdot \frac{\vec{r}}{r} = \frac{-GM_E m_{sv}}{r^3}\vec{r}$$

  - Earth:

$$M_E\ddot{\vec{r}}_E = \frac{GM_E m_{sv}}{r^2} \cdot \frac{\vec{r}}{r} = \frac{GM_E m_{sv}}{r^3}\vec{r}$$

Newton vs. Kepler



- Taking the difference in the two equations:

$$M_E m_{sv} \ddot{\vec{r}}_s - M_E m_{sv} \ddot{\vec{r}}_E = \frac{-G M_E m_{sv}^2}{r^3}\vec{r} - \frac{G M_E^2 m_{sv}}{r^3}\vec{r}$$

- Provides the relative position vector:

$$M_E m_{sv} \ddot{\vec{r}} = \frac{-G}{r^3}\vec{r}\left(M_E m_{sv}^2 + M_E^2 m_{sv}\right)$$

$$\ddot{\vec{r}} = \frac{-G}{r^3}\vec{r}\left(M_E + m_{sv}\right)$$

Newton vs. Kepler

$$\ddot{\vec{r}} + \frac{GM_{tot}}{r^3}\vec{r} = 0$$

- 6th order non-linear homogeneous differential equation
  - Requires 6 Initial Conditions
    - $\vec{r}(o)$ and $\dot{\vec{r}}(o)$
- The solution to the differential equations results in Kepler's 3 Laws of orbits

1) Elliptical Motion

2) Motion is faster when closer to the orbiting body (i.e. Earth for SVs)

3) $t_{orbit}^2 = kd_{avg}^3$

Position of the SV in orbital plane

Orbits (2 of 5)

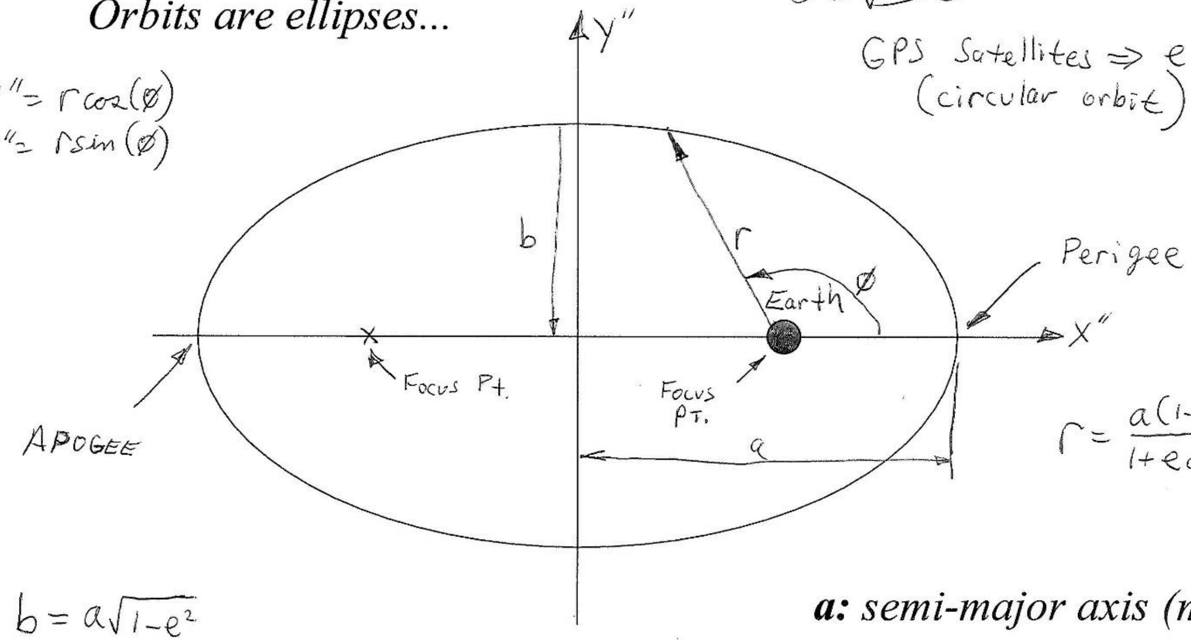
$$T_p = \frac{2\pi}{\sqrt{\mu_e}} a^{3/2}$$

↳ Period of the orbit

Orbits are ellipses...

$x'' = r\cos(\phi)$
$y'' = r\sin(\phi)$

GPS Satellites ⇒ $e \sim$ small
(circular orbit)



$$r = \frac{a(1-e^2)}{1+e\cos(\phi)}$$

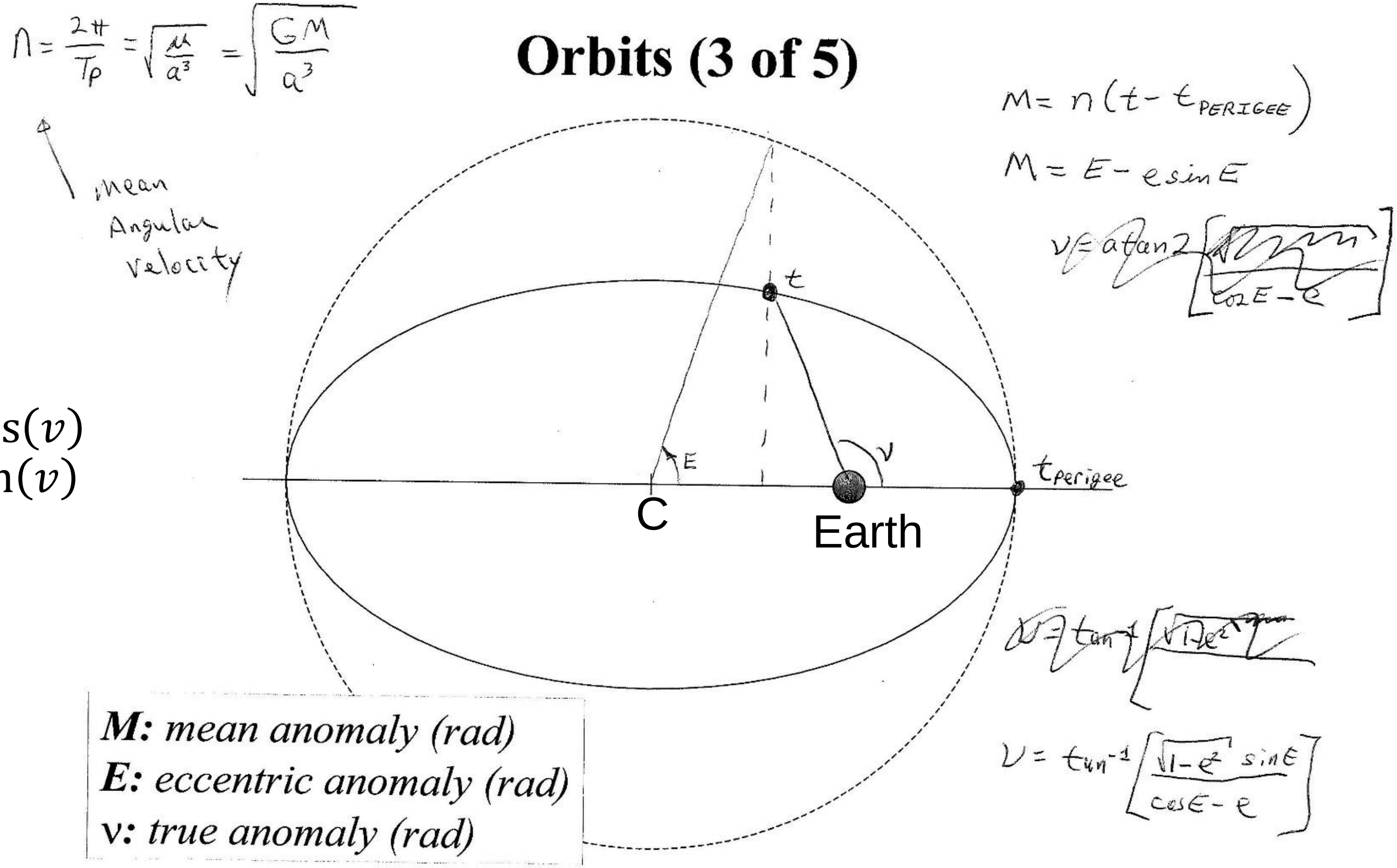
$b = a\sqrt{1-e^2}$

e ⇒ eccentricity of the orbit

a: *semi-major axis (m)*
e*: eccentricity*
n*: mean motion (rad/s)*

$$e^2 = a^2 - b^2$$

Definitions of the orbital frame



$$x = |r|\cos(v)$$
$$y = |r|\sin(v)$$

$$|r| = \frac{a(1-e^2)}{1+e\cos(v)} = a\big(1 - e\cos(E)\big)$$

$$v = tan^{-1}\left(\frac{\sin(E)\sqrt{1-e^2}}{\cos(E) - e}\right)$$

Position Variables in the Orbital Plane

- Mean angular velocity:
  - $n = \frac{2\pi}{T} = \frac{GM}{a^3}$
  - where *M* is the mean anomaly
    - The angle from perigee and an SV at constant velocity in a circular orbit with the same focus and period as the real SV (i.e. they cross at perigee and apogee at the same time)

$$M_{SV} = E_{SV_{circular\,orbit}}$$

$$M = n\left(t - t_{perigee}\right) = E - esin(E)$$

  - Must solve for *M* iteratively until $\Delta E < 1 \times 10^{-12}$

$$E = M + esin(E)$$

SV Velocities

- Taking the derivative with respect to time results in:

$$\dot{M} = \dot{E}(1 - \cos(E)) = n$$

$$\dot{r}_T = \dot{R}_3(\theta) r_I + R_3(\theta)\dot{r}_I$$

$$\dot{r}_x = \frac{-nasin(E)}{1-ecos(E)} \qquad \dot{r}_y = \frac{nacos(E)\sqrt{1-e^2}}{1-ecos(E)}$$

- Taking the derivative again results in:

$$\ddot{\vec{r}} = \frac{-GM_{tot}}{r^3}\vec{r}$$

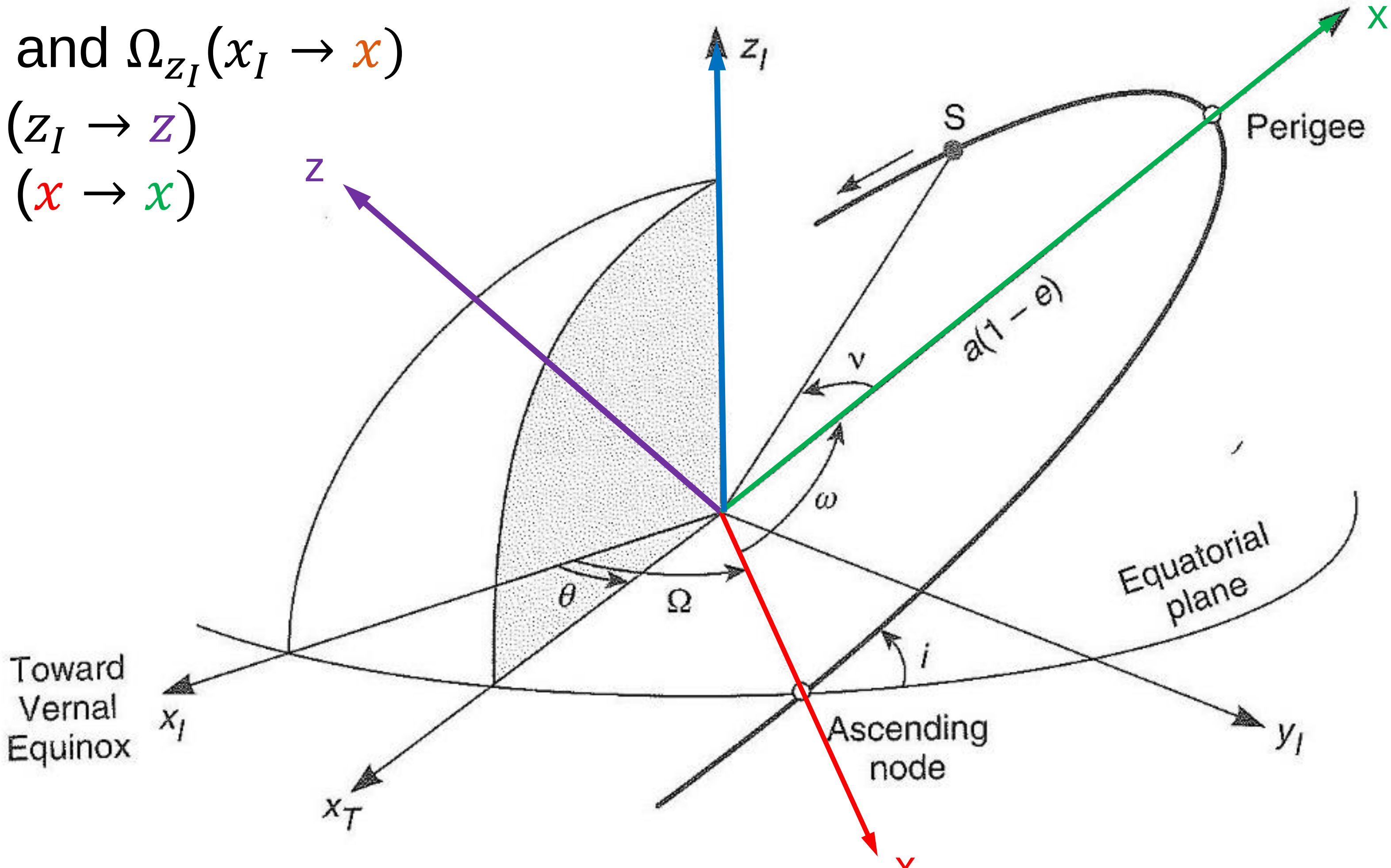
Rotation Matrices

- We have to move the SV positions from their orbital frame to the Earth Center Earth Frame (ECEF)
- This requires rotating the position from one frame to the other
  - This is done through rotation matrices
  - NOTE: Order of rotations is ***critical*** (i.e. the order changes the answer)
    - Ex: roll 90, pitch 90, yaw 90 vs. yaw 90, roll 90, pitch 90
    - A couple of good resources:
      - http://www.chrobotics.com/library/understanding-euler-angles
      - https://phas.ubc.ca/~berciu/TEACHING/PHYS206/LECTURES/FILES/euler.pdf

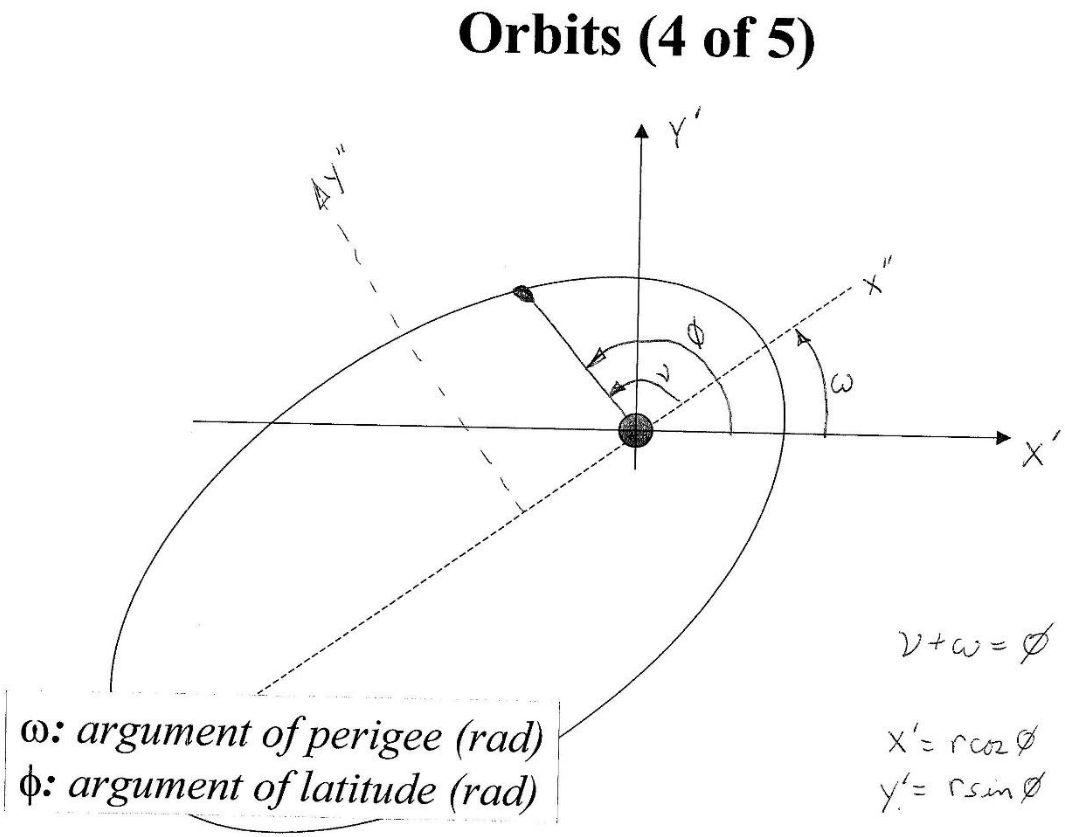
Rotations from ECEF to Orbit Frame

- From Inertial to Body Frame

1st: θ_{z_I} and $\Omega_{z_I}(x_I \rightarrow x)$
2nd: i_x $(z_I \rightarrow z)$
3rd: ω_z $(x \rightarrow x)$



Rotate the Position about z-axis



$\nu + \omega = \phi$

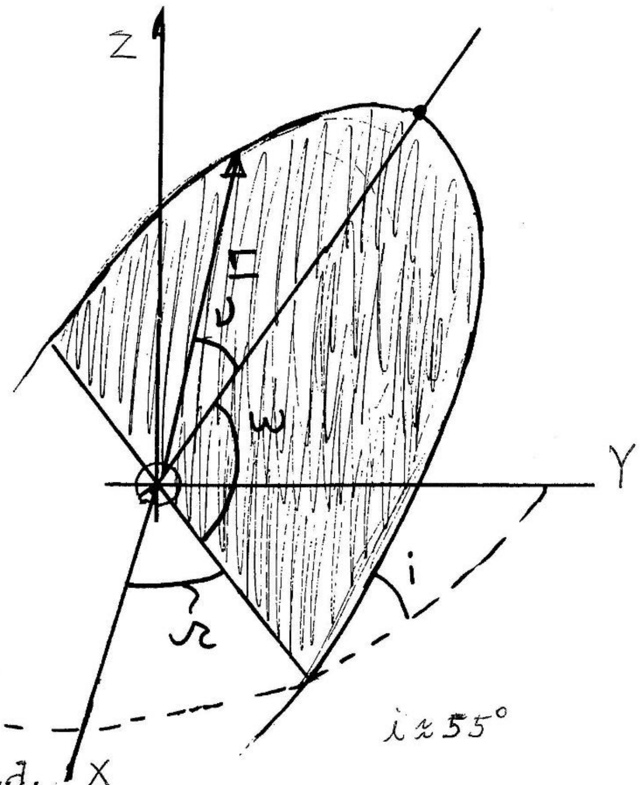
$x' = r\cos\phi$

$y' = r\sin\phi$

Rotation about x-axis

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} \ \end{bmatrix}$$

Orbits (5 of 5)



i: inclination (rad)

Ω: longitude of ascending node (rad)

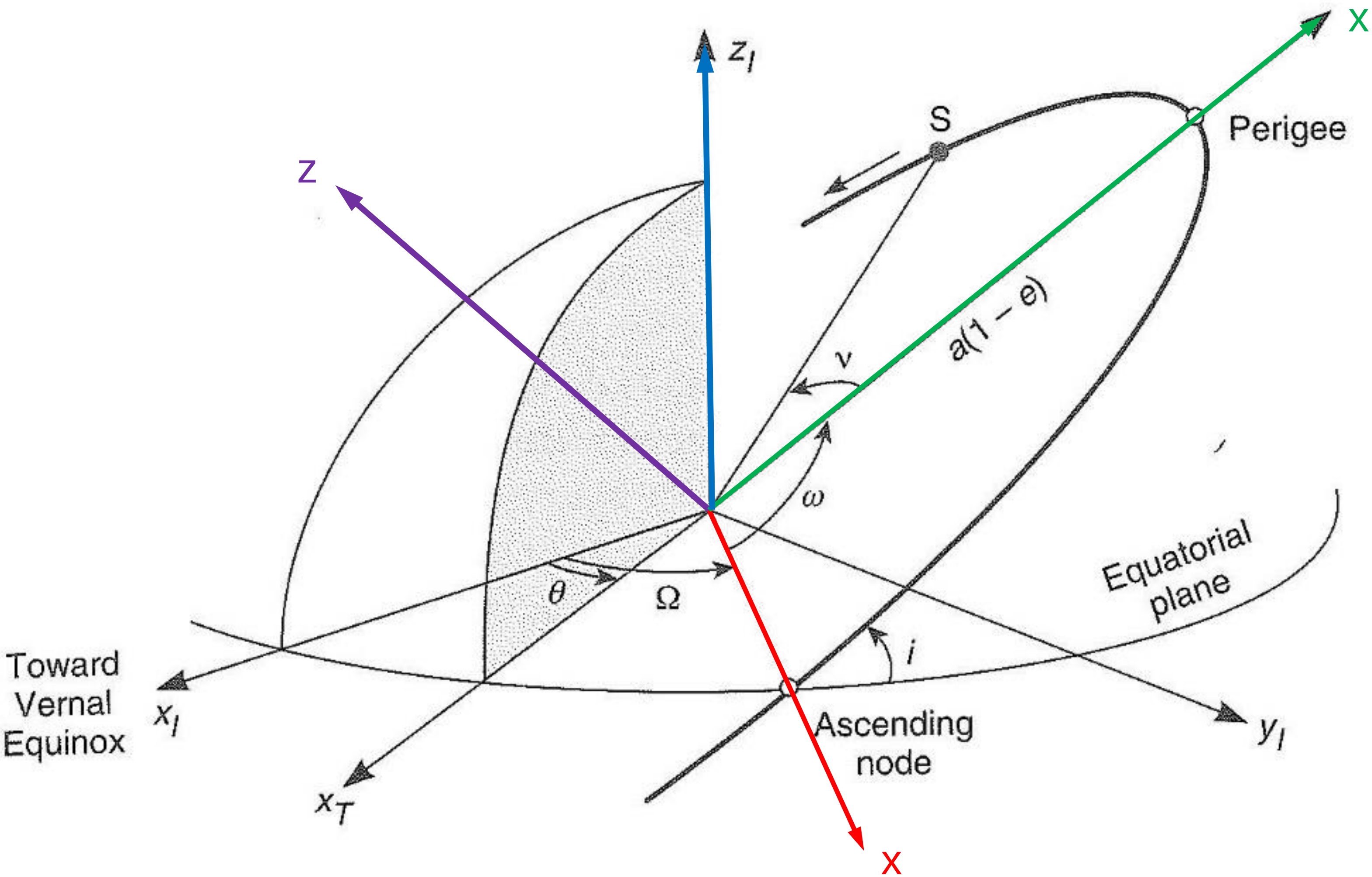
Position Translation (w/ Rotation Matrices)

- To calculate the position in the orbital frame from the inertial (ECEF) frame is done by:
  - Spin by Ω deg about z axis
    - This rotates the x-y axis around the earth
  - Then spin by i degrees about the x axis
    - This rotates the y-z axis to the orbit inclination
  - Finally spin by ω degrees about the z axis
    - This rotates the x-y axis about the earth to place the ellipse “centered” correctly

$$\vec{r} = R_3(\omega)\, R_1(i)\, R_3(\Omega)\vec{r}_I$$

Rotations from ECEF to Orbit Frame

$$\vec{r} = R_3(\omega)\ R_1(i)\ R_3(\Omega)\vec{r}_I$$



Rotation Matrices

- In reality, the SV position is defined in the orbital plane and we must calculate the position in the ECEF Frame
- Using properties of rotation matrix inverses

$$R^{-1}(\theta) = R(-\theta) = R^{T}(\theta)$$

  – results in: $\vec{r}_I = R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{r}$
- Then rotating to the Greenwich Sidereal time:

$$\vec{r}_T = R_3(\theta)\vec{r}_I$$

  – Note this last rotation is about the same axis as the RAAN angle. GPS definitions combine these two rotations!

GPS SV Position Calculation

- GPS also calculates the position from the ascending node:

$$\phi = \omega + v$$

  – Therefore we do not have to do the last rotation about the z axis.

- Position is then calculated as:

$$x = r\cos(\phi)$$
$$y = r\sin(\phi)$$

- Rotating the position from the orbital frame into inertial frame:

$$\vec{r}_I = R_3(-\Omega)R_1(-i)\vec{r}$$

GPS SV Position Rotation

- Rotating the position into inertial frame:

$$\vec{r}_I = R_3(-\Omega)R_1(-i)\vec{r}$$

- As mentioned previously, GPS uses Longitude of Ascending Node (LAN) which combines the Right Ascension of Ascending Node (RAAN) and the Greenwich Apparent Sidereal Time (GAST) rotations as:

$$\Omega = \Omega_{LAN}(t) = \Omega_{RAAN} - \theta_{GAST}(t)$$

  – This makes it easier to go to WGS84 ECEF Frame

GPS Position Rotations

- Calculating the SV position in ECEF:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_3(-\Omega)R_1(i)\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c\Omega & -s\Omega & 0 \\ s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & -si \\ 0 & si & ci \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$= \begin{bmatrix} c\Omega & -s\Omega & 0 \\ s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y'ci \\ y'si \end{bmatrix} = \begin{bmatrix} x'cos(\Omega) - y'sin(\Omega)\cos(i) \\ x'sin(\Omega) + y'cos(\Omega)\sin(i) \\ y'sin(i) \end{bmatrix}$$

  - Where

$$x' = rcos(v + \omega)$$
$$y' = rsin(v + \omega)$$
$$z' = 0$$
$$\Omega = \Omega_{LAN}$$

Orbit Pertubations

- Orbits are perturbed
  - Rocket firing interventions
  - Non-central (uniform) gravitational force field
  - Equatorial bulge
    - Produced torque on SV
    - Harmonic pertubations (twice per orbit)
  - Gravity of Sun and Moon
  - Solar radiation pressure

$$\ddot{\vec{r}} = -\frac{GM_{tot}}{r^3}\vec{r} + F_{dist}(r, \dot{r}, t)$$

$$\frac{GM_{tot}}{r^3}\vec{r} \gg F_{dist}(r, \dot{r}, t)$$

GPS SV Positions

GPS Ephemeris, cont'd.

$$\boxed{\overline{X}^{(k)} = \overline{X}^{(k)}_{Broadcast} + \underbrace{d\overline{X}^{(k)}}_{\text{Error} \rightsquigarrow \text{small}}}$$

But again, GPS does not broadcast its position but rather ephemerides and ephemeris correction terms (curve fits) to calculate the correct SV position (from Kepler orbital mechanics)

GPS Ephemeris

GPS Ephemeris

- "Ephemeris" = Orbit data
- "Ephemerides" = Individual parameters of orbit

You provide: t

GPS provides: t_{oe}

...and nominal ephemerides: $\sqrt{a}, e, M_0, \omega_0, i_0, \Omega_0$

...and pertubation effects:
(1) Non-spherical Earth
(2) Tidal effects
(3) Solar radiation pressure

$\Delta n, (IDOT), \dot{\Omega},$ Secular Pertubation

$C_{u\cos}, C_{u\sin}, C_{r\cos}, C_{r\sin}, C_{i\cos}, C_{i\sin}$ Harmonic Pertubation

SV Position Subtilties

- Note that “t” is transmit time (i.e. time at SV transmission), so it must be corrected for transit time. This is done by taking the range/c.
  – You can use the corrected pseudorange/c
    - Will have some small error
  – Or you must solve for the SV positions iteratively with your position to calculate exact transit time.
- Additionally you ***may*** want to account for the fact that the earth has rotated during the transit time
  – Some code (including what I share on the website) does this.
  – Blue book and Akos SV calculator do not.

SV Calculation Equation

Table 8 Elements of ephemeris model equations[1]

$\mu = 3.986005 \times 10^{14}\ \mathrm{m^3/s^2}$	WGS-84[5] value of the Earth's universal gravitational parameter
$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/s	WGS-84[5] value of the Earth's rotation rate
$A = (\sqrt{A})^2$	Semimajor axis
$n_0 = \sqrt{\mu/A^3}$	Computed mean motion–rad/s
$t_k = t - t_{oe}$[a]	Time from ephemeris reference epoch
$n = n_0 + \Delta n$	Corrected mean motion
$M_k = M_o + nt_k$	Mean anomaly
$\pi = 3.1415926535898$	GPS standard value for π
$M_k = E_k - e \sin E_k$	Kepler's equation for the eccentric anomaly E_k (may be solved by iteration), rad
$\nu_k = \tan^{-1}\left\{\dfrac{\sin \nu_k}{\cos \nu_k}\right\} = \tan^{-1}\left\{\dfrac{\sqrt{1-e^2}\sin E_k/(1 - e\cos E_k)}{(\cos E_k - e)/(1 - e \cos E_k)}\right\}$	True anomaly ν_k as a function of the eccentric anomaly
$E_k = \cos^{-1}\left\{\dfrac{e + \cos \nu_k}{1 + e \cos \nu_k}\right\}$	Eccentric anomaly
$\Phi_k = \nu_k + \omega$	Argument of latitude
$\delta u_k = C_{us} \sin 2\Phi_k + C_{uc} \cos 2\Phi_k$	Argument of latitude correction — Second harmonic perturbations
$\delta r_k = C_{rs} \sin 2\Phi_k + C_{rc} \cos 2\Phi_k$	Radius correction — Second harmonic perturbations
$\delta i_k = C_{is} \sin 2\Phi_k + C_{ic} \cos 2\Phi_k$	Inclination correction — Second harmonic perturbations
$u_k = \Phi_k + \delta u_k$	Corrected argument of latitude
$r_k = A(1 - e \cos E_k) + \delta r_k$	Corrected radius
$i_k = i_0 + \delta i_k + (\mathrm{IDOT})\, t_k$	Corrected inclination
$x_k' = r_k \cos u_k$ $y_k' = r_k \sin u_k$	Satellite position in orbital plane
$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e)\, t_k - \dot{\Omega}_e\, t_{oe}$	Corrected longitude of ascending node
$x_k = x_k' \cos \Omega_k - y_k' \cos i_k \sin \Omega_k$ $\cancel{y_k = y_k' \sin \Omega_k + y_k' \cos i_k \cos \Omega_k}$ $z_k = y_k' \sin i_k$	Satellite position in Earth-centered, Earth-fixed coordinates

[a] t is GPS system time at time of transmission; i.e., GPS time corrected for transit time (range/speed of light). Furthermore, t_k shall be the actual total time difference between the time t and the epoch time t_{oe} and must account for beginning or end of week crossovers. That is, if t_k is greater than 302,400 s, subtract 604,800 s from t_k. If t_k is less than 302,400 s, add 604,800 s to t_k.

Note: You must iterate to solve Kepler's equation for the eccentric anomaly, (i.e., to solve for E given M). There are many interesting ways to do this, but for this exercise, simply iterate on the equation *E=M+e*sin(E)*. Start with *E=M* on the right side of the equation, solve for E on the left side, then plug that value back into the right side of the

SV Clock Data Corrections

GPS Satellite Clock

Error

$$\boxed{B^{(k)} = B^{(k)}_{Broadcast} + dB^{(k)}}$$

Hope Error is small – but its Not Because of SA

Clock Bias

Clock Rate

TTRANSMISSION (Based on GPS Time)

$$B^{(k)}_{Broadcast} = af_0 + af_1(\boldsymbol{Ttr} - Toc) + af_2(\boldsymbol{Ttr} - Toc)^2 + \Delta T_{rel} + T_{gd}$$

From Satellite

$\dot{f} \approx 0$

TOF CORRECTION

Based ON ORBIT

Don't forget that this term must be used to correct the pseudoranges in the PVT solution

Ephemeris Updates

- Ephemeris are updated every 2 hours
- Issue of Data Ephemeris (IODE)
  - Change in IODE indicates an update to the ephemeris
- Ephemeris are good for 4 hours
  - Will maintain GPS spec for up to 4 hours
- Ephemeris refers to the group of data (each are called ephemerides)
- Must check for time rollover of $t - t_{oe}$ at beginning/end of week

Broadcast Ephemeris

AUBURN UNIVERSITY

So How Do We Get Broadcast Parameters in Real Life?

- **Navigation message**: Data stream broadcast from each satellite
- You can only get the data from satellites you are tracking
- Overlaid on GPS code (the "chips")
- GPS C/A code repeats 20 times per bit
- 50 bits/second
- 1500 bits = 1 "frame" --> 1 frame = 30 seconds
- Frames "repeat" every 30 seconds

NOTE: Takes 30 seconds to receive all the ephemeris to compute the SV potions (but after 30 seconds, the data is good to be used for 4 hours!)

Overview of Sub-frames

Frames and Sub-Frames (1 of 5)

Ten, 30 bit words forming a six second subframe

The five subframes form a 30 second frame with 1500 bits total.

Subframe Number			
1	TLM	HOW	Block 1 - Clock Correction
2	TLM	HOW	Block 2 - Ephemeris
3	TLM	HOW	Block 3 - Ephemeris (Cont.)
4	TLM	HOW	Block 4 - Message
5	TLM	HOW	Block 5 - Almanac (25 frames are required for the complete almanac)

(from Global Positioning System: Theory and Applications by AIAA)

TLM and HOW

- TLM begins with an 8 bit synchronization pattern
  - 10001011 (0x8B)
  - Occurs every 6 seconds
- HOW is the 17 MSB of the 19 bit Time of Week (TOW) count
  - 6 seconds of resolution
- GPS Time is 29 bits
  - 10 bits for week (1024)
  - 19 bits for TOW (1.5 second increments)

Overview of Sub-frames

Frames and Sub-Frames (2 of 5)

- Subframe = 6 seconds (300 bits)
- 5 subframes per frame
- Subframes 1-3: "Repeat" every 30 seconds
- Subframes 4-5:
  - 25 "pages" for each, repeating after page number 25.
  - Pages increment each 30 seconds
  - Thus, it takes 25 x 30 seconds = 12.5 minutes to *guarantee* reception of all 25 pages for subframes 4 & 5 (assuming continuous navigation data signal)

Overview of Sub-frames

Frames and Sub-Frames (3 of 5)

- **Subframe 1:** Info. and clock parameters *for satellite being tracked:*

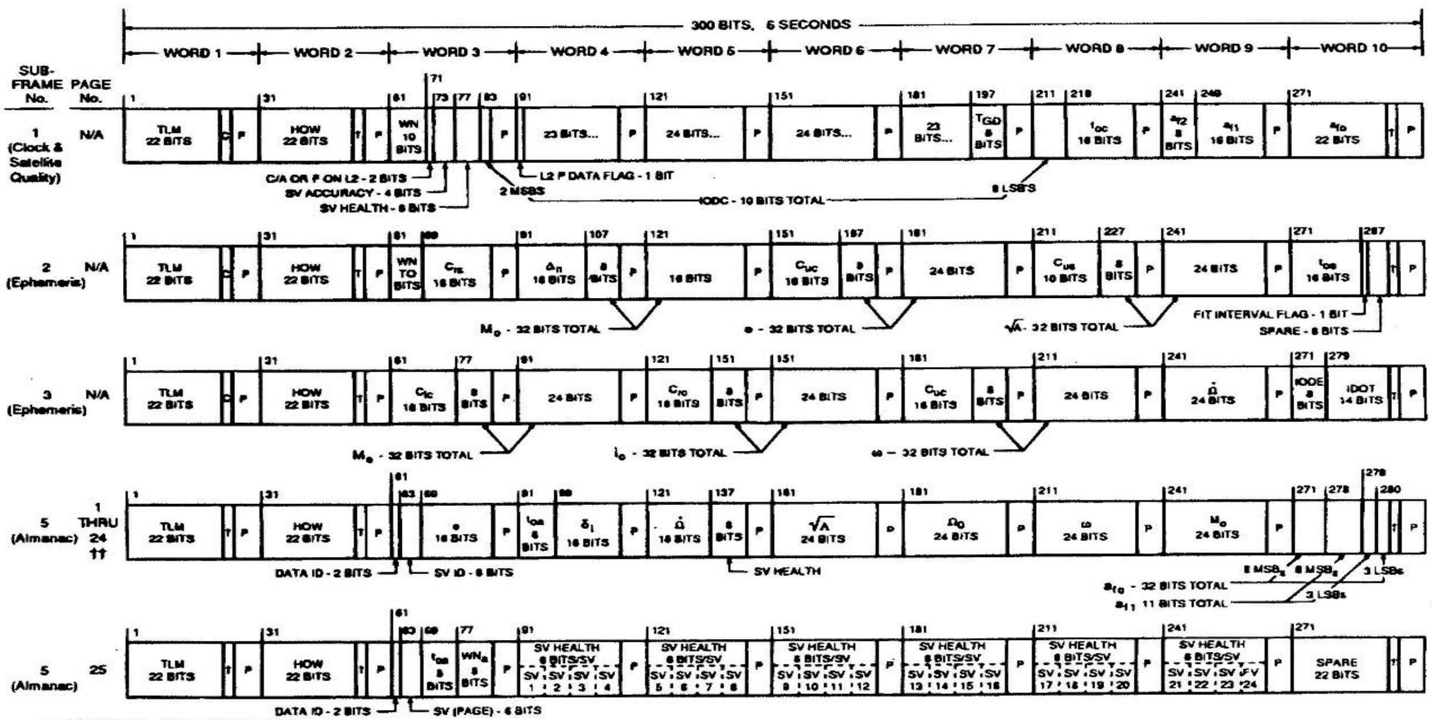
- **Subframe 2:** Epemerides *for satellite being tracked.*

- **Subframe 3:** More ephemerides *for satellite being tracked.*
- **Subframe 4:** Information *for GPS system* or **almanac** *for 1 satellite (not necessarily the satellite being tracked).*

- **Subframe 5:** More **almanacs** *for 1 satellite (not necessarily the satellite being tracked).*

Sub-frame Details

Frames and Sub-Frames (4 of 5)



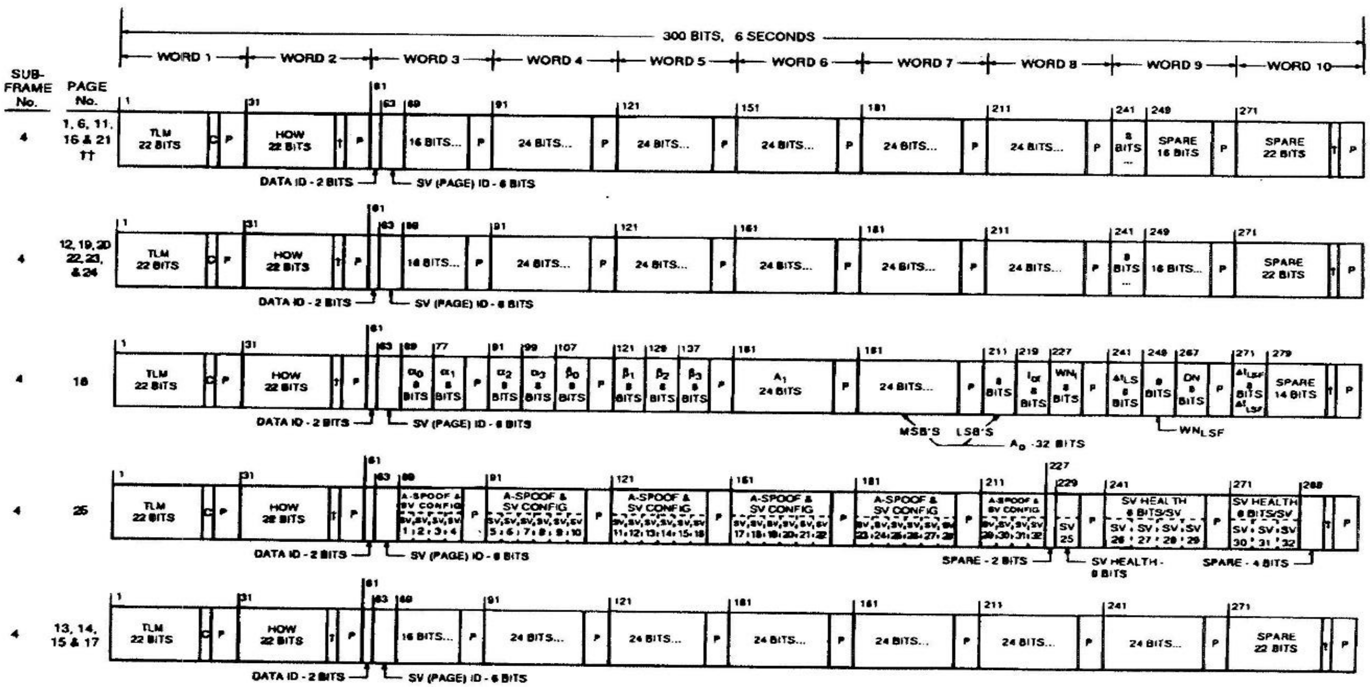
• • • RESERVED - DISCUSSED LATER
† = TWO NONINFORMATION BEARING BITS USED FOR PARITY COMPUTATION (SEE PARITY CHECK ALGORITHM)
†† = PAGES 2, 3, 4, 5, 7, 8, 9, AND 10 OF SUBFRAME 4 HAVE SAME FORMAT AS PAGES 1 THROUGH 24 OF SUBFRAME 5

P = SIX PARITY BITS
C = TLM BITS 23 AND 24 ARE RESERVED

(from Global Positioning System: Theory and Applications by AIAA)

Sub-frame Details

Frames and Sub-Frames (5 of 5)



••• RESERVED - DISCUSSED LATER
† = TWO NONINFORMATION BEARING BITS USED FOR PARITY COMPUTATION (SEE PARITY CHECK ALGORITHM)
†† = PAGES 2, 3, 4, 5, 7, 8, 9, AND 10 OF SUBFRAME 4 HAVE SAME FORMAT AS PAGES 1 THROUGH 24 OF SUBFRAME 5
P = SIX PARITY BITS
C = TLM BITS 23 AND 24 WHICH ARE RESERVED

(from Global Positioning System: Theory and Applications by AIAA)

Example Ephemeris

Example: Navigation data for PRN 6

TIME (sec)	*SUBFRAME (page)*	*MESSAGE*
0	*1*	*Info, clock for PRN 6*
6	*2*	*Ephemeris for PRN 6*
12	*3*	*More ephemeris for PRN 6*
18	*4(18)*	*Ionosphere, week number, etc.*
24	*5(18)*	*Almanac for PRN 2*
30	*1*	*Info, clock for PRN 6*
36	*2*	*Ephemeris for PRN 6*
42	*3*	*More ephemeris for PRN 6*
48	*4(19)*	*More GPS info....*
54	*5(19)*	*Almanac for PRN 3*

Reference Guide

- GPS Interface Specifications:
  - IS-GPS-200D (revised 2006)

https://www.gps.gov/technical/icwg/IS-GPS-200D.pdf

  - Appendix II (pp 65-136) provides details on broadcast data
    - Table 20-IV (pp 97-98) provides the SV position calculation details

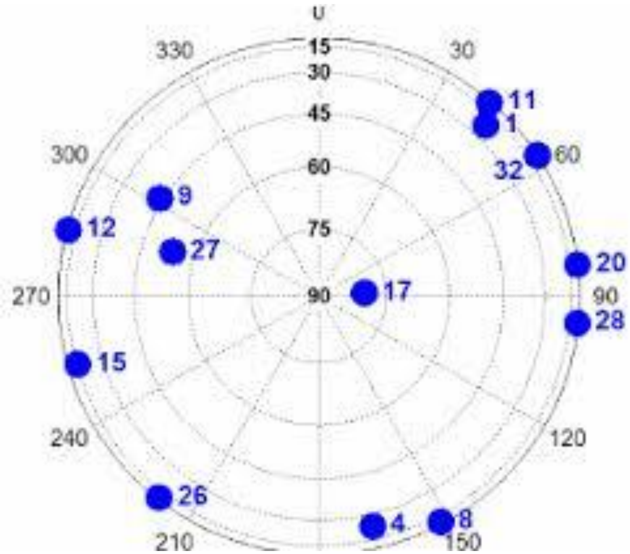
Ephemeris Data Repositories

- https://www.igs.org/products#precise_orbits
- https://urs.earthdata.nasa.gov/oauth/authorize?client_id=gDQnv1IO0j9O2xXdwS8KMQ&response_type=code&redirect_uri=https%3A%2F%2Fcddis.nasa.gov%2Fproxyauth&state=aHR0cDovL2NkZGlzLm5hc2EuZ292L2FyY2hpdmUvZ25zcy9wcm9kdWN0cy8
- https://cddis.nasa.gov/Data_and_Derived_Products/GNSS/orbit_products.html
- https://cddis.nasa.gov/Data_and_Derived_Products/GNSS/broadcast_ephemeris_data.html
- https://www.ngs.noaa.gov/orbits/

Some of the above contain “precise” ephemeris (i.e. correct ephemeris)

GNSS Planning Tools (and Skyplots)

- Sky plots show the satellite locations with respect to the user in elevation (from the horizon) and azimuth (from north)



https://www.gnssplanning.com

http://gnssmissionplanning.com

https://www.mathworks.com/matlabcentral/fileexchange/25557-sky-plot-3d