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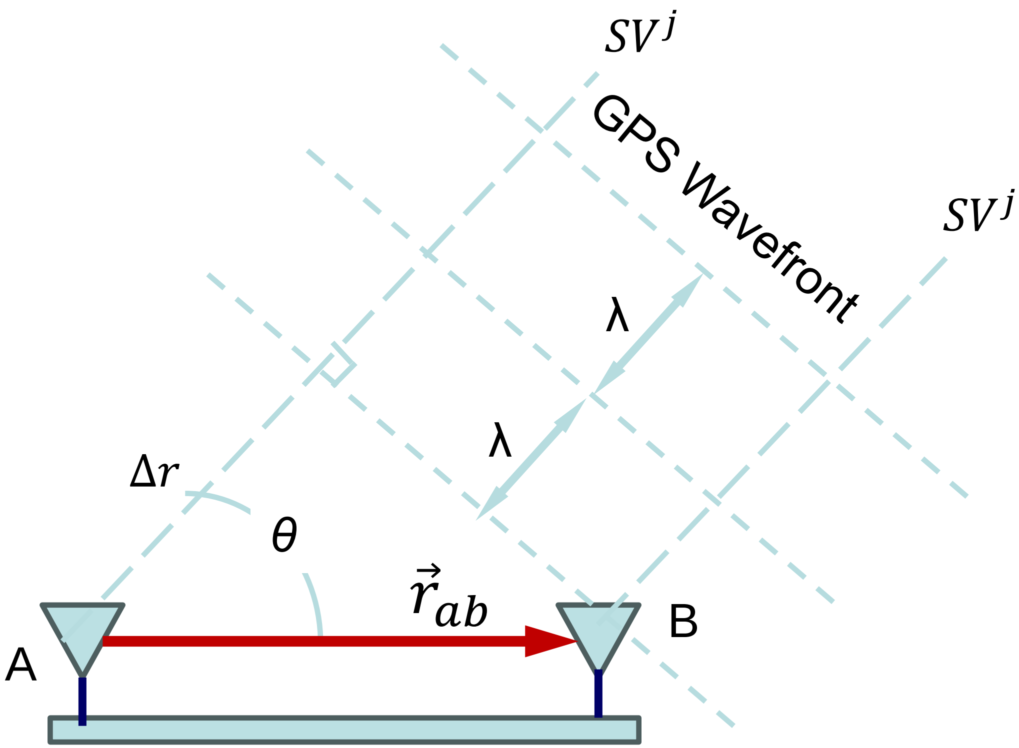
Differential GPS

MECH 5970/6970
Fundamentals of GPS

Relative Positioning

- Determination of the baseline vector between a known receiver location and arbitrary receiver location
  - If receivers are in close proximity (50km), they are subjected to very similar errors
  - Differencing measurements from receivers removes errors, providing accurate baseline measurement
    - Single differencing – removes atmospheric errors and satellite clock biases
    - Double differencing – additionally removes receiver clock bias
    - Carrier triple differencing – removes cycle slip effects

GPS Measurement Models



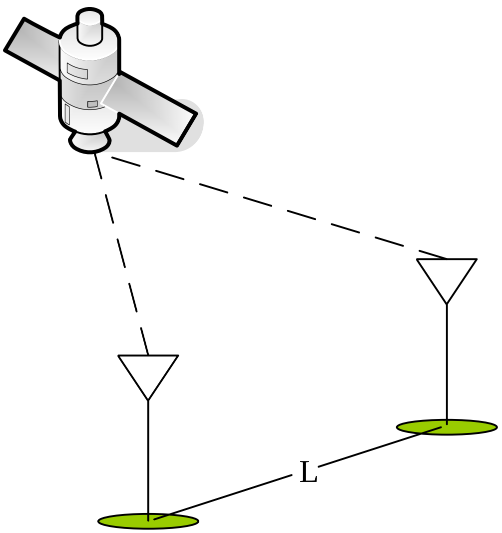
$$\Delta r = |\vec{r}_{ab}| cos(\theta)$$

$$\Delta r^j = \rho_A^j - \rho_B^j = \Delta \rho_{AB}^j$$

$$\rho_A^j = r_A^j + c\left(\delta t_A - \delta t^j\right) + I^j + T^j + v_\rho^j$$
$$\Phi_A^j = r_A^j + c\left(\delta t_A - \delta t^j\right) + \lambda N_A^j - I^j + T^j + v_\Phi^j$$

Single Differencing

- Involves differencing receiver measurements from two receivers, *A* and *B*, using a common satellite, *j*



Measurement model:

$$\rho_A^j = r_A^j + c\left(\delta t_A - \delta t^j\right) + I^j + T^j + v_\rho^j$$
$$\rho_B^j = r_B^j + c\left(\delta t_B - \delta t^j\right) + I^j + T^j + v_\rho^j$$
$$\Phi_A^j = r_A^j + c\left(\delta t_A - \delta t^j\right) + \lambda N_A^j - I^j + T^j + v_\Phi^j$$
$$\Phi_B^j = r_B^j + c\left(\delta t_B - \delta t^j\right) + \lambda N_B^j - I^j + T^j + v_\Phi^j$$

Differencing measurements from two receivers yields:

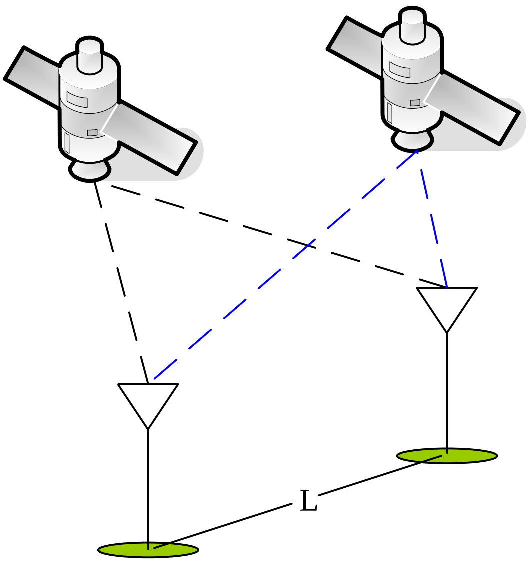
$$\Delta\rho_{AB}^j = \rho_A^j - \rho_B^j = r_A^j - r_B^j + c(\delta t_A - \delta t_B) + \Delta v_\rho^j$$
$$\Delta\Phi_{AB}^j = r_A^j - r_B^j + c(\delta t_A - \delta t_B) + \lambda(N_A^j - N_B^j) + \Delta v_\Phi^j$$

Single differenced solution:

$$\Delta\rho_{ab}^j = \Delta r^j + ct_{ab} + \epsilon_{\Delta\rho} = \vec{r}_{ab}^{\,j} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{ab}^j = \Delta r^j + \lambda N_{ab}^j + ct_{ab} + \epsilon_{\Delta\Phi} = \vec{r}_{ab}^{\,j} + \lambda N_{ab}^j + ct_{ab} + \epsilon_{\Delta\Phi}$$

Double Differencing

- Involves differencing single difference measurements from two receivers, *A* and *B*, between two common satellites, *j* and *k*



Two single differences are calculated:

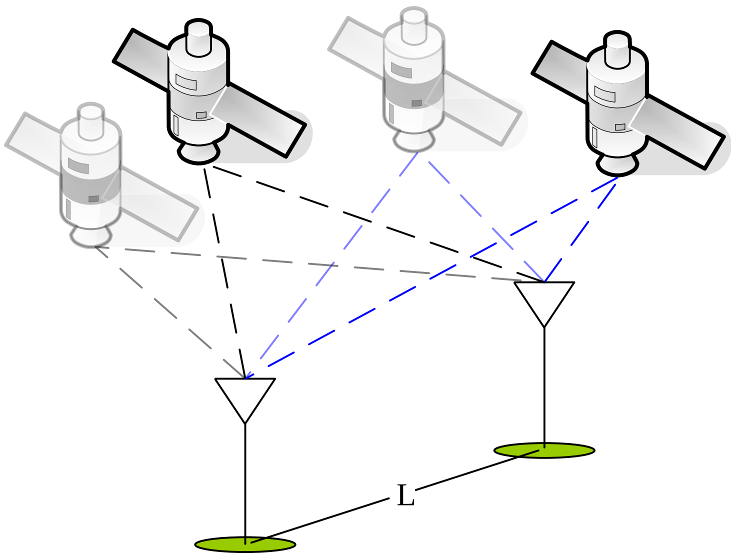
$$\Delta\rho_{ab}^{j} = \Delta r^{j} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\rho_{ab}^{k} = \Delta r^{k} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{ab}^{j} = \Delta r^{j} + \lambda N_{ab}^{j} + ct_{ab} + \epsilon_{\Delta\Phi}$$
$$\Delta\Phi_{ab}^{k} = \Delta r^{k} + \lambda N_{ab}^{k} + ct_{ab} + \epsilon_{\Delta\Phi}$$

Differencing two single differences yields:

$$\nabla\Delta\rho_{ab}^{jk} = \vec{r}_{ab}^{jk} + \epsilon_{\nabla\Delta\rho}$$
$$\nabla\Delta\Phi_{ab}^{jk} = \vec{r}_{ab}^{jk} + \lambda N_{ab}^{jk} + \epsilon_{\nabla\Delta\Phi}$$

Carrier Triple Differencing

- Differences the double differences at multiple epochs of time



Two double differences are calculated at different periods:

$$\nabla\Delta\rho_{ab}^{jk}(t_0) = \vec{r}_{ab}^{jk}(t_0) + \epsilon_{\nabla\Delta\rho}$$
$$\nabla\Delta\rho_{ab}^{jk}(t_1) = \vec{r}_{ab}^{jk}(t_1) + \epsilon_{\nabla\Delta\rho}$$
$$\nabla\Delta\Phi_{ab}^{jk}(t_0) = \vec{r}_{ab}^{jk}(t_0) + \lambda N_{ab}^{jk}(t_0) + \epsilon_{\nabla\Delta\Phi}$$
$$\nabla\Delta\Phi_{ab}^{jk}(t_1) = \vec{r}_{ab}^{jk}(t_1) + \lambda N_{ab}^{jk}(t_1) + \epsilon_{\nabla\Delta\Phi}$$

The differenced double differences are:

$$\nabla\Delta\rho_{ab}^{jk}(t_1) - \nabla\Delta\rho_{ab}^{jk}(t_0) = \Delta\vec{r}_{ab}^{jk}$$
$$\nabla\Delta\Phi_{ab}^{jk}(t_1) - \nabla\Delta\Phi_{ab}^{jk}(t_0) = \Delta\vec{r}_{ab}^{jk}$$

Therefore (assuming no cycle slip):

$$\Delta\nabla\Delta\,\Phi_{ab}^{jk} = \Delta\nabla\Delta\rho_{ab}^{jk}$$

Single Difference Position Equations

- Single Difference
- $\Delta r^j = \left(uv_x^j\right)\Delta x_{ab} + \left(uv_y^j\right)\Delta y_{ab} + \left(uv_z^j\right)\Delta z_{ab}$
- $\Delta r^j = \vec{r}_{ab}^{\,j} = \begin{bmatrix} uv_x^j & uv_y^j & uv_z^j \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab}$
- Single Difference Position Solution:
  – Requires 4 common SVs to solve

$$\Delta\rho_{ab}^j = \begin{bmatrix} uv_x^j & uv_y^j & uv_z^j \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab} + c\delta t_{ab} + \epsilon_{\Delta\rho}$$

$$\Delta\Phi_{ab}^j = \begin{bmatrix} uv_x^j & uv_y^j & uv_z^j \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab} + c\delta t_{ab} + \lambda N_{ab}^j + \epsilon_{\Delta\Phi}$$

Double Difference Position Equations

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- Double Difference

$$\Delta r^j - \Delta r^k = \begin{bmatrix} uv_x^j & uv_y^j & uv_z^j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab} - \begin{bmatrix} uv_x^k & uv_x^k & uv_x^k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab}$$

- Double Difference Position Solutions:
  - Still requires 4 common SVs to solve (need one common SV to form 3 DD measurements):

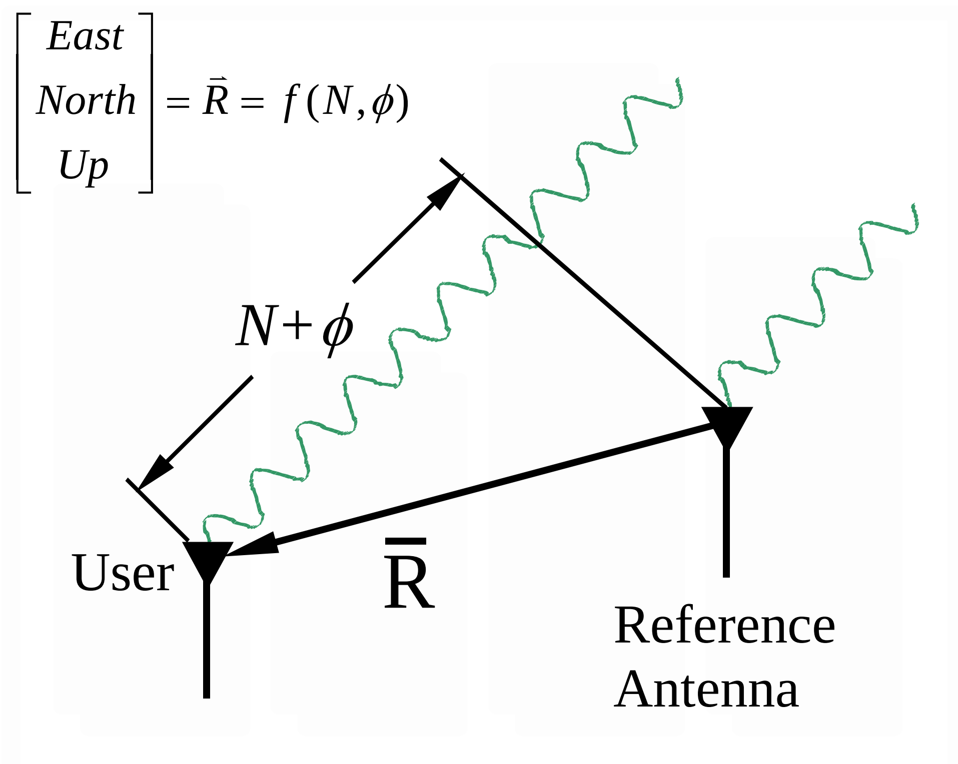
$$\nabla\Delta\rho_{ab}^{jk} = \begin{bmatrix} \Delta uv_x^{jk} & \Delta uv_y^{jk} & \Delta uv_z^{jk} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab} + \epsilon_{\nabla\Delta\rho}$$

$$\nabla\Delta\Phi_{ab}^{jk} = \begin{bmatrix} \Delta uv_x^{jk} & \Delta uv_y^{jk} & \Delta uv_z^{jk} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ab} + \lambda N_{ab}^{jk} + \epsilon_{\nabla\Delta\Phi}$$

Time Difference Carrier Phase (TDCP)

- Somewhat related to the triple difference is what is known as the time difference carrier phase
  - This can be done from multiple measurements
    - Original Signal
      - Contains drift due to change in atmospheric errors
    - Single Difference
      - Contains clock drift
    - Double Difference

Carrier Phase DGPS (RTK)



- local reference station required
- solve for integer ambiguity
- track carrier phase
- phase at reference antenna is broadcast to user
- positioning software calculates $\vec{R}$
- 3-D accuracy* = 2 cm

*Actual depends on baseline length (1 cm + 1 ppm)

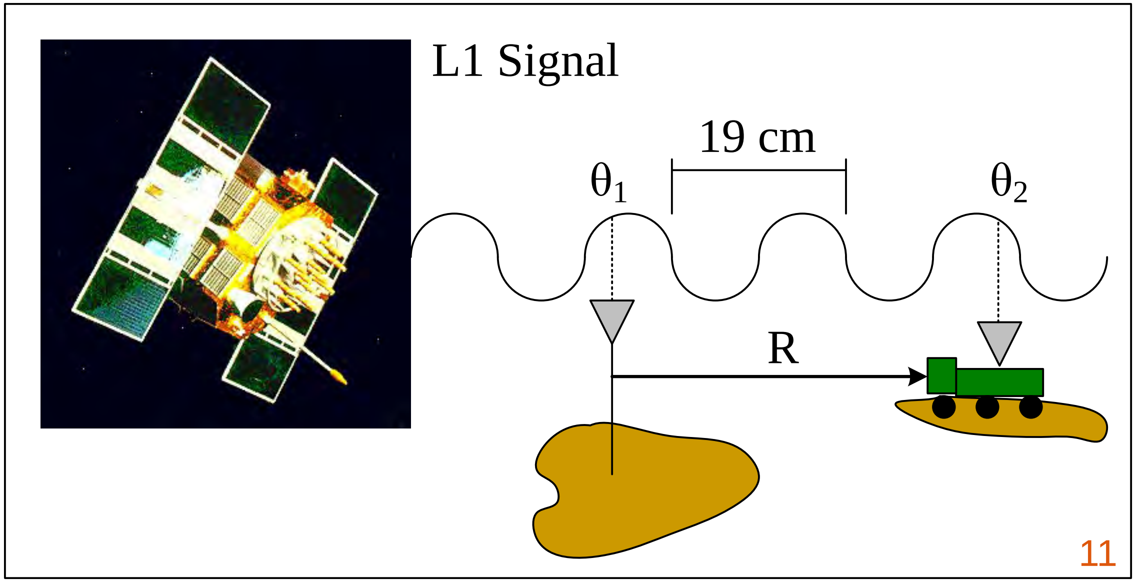
More on RTK

- RTK – Real Time Kinematic GPS
- RTK GPS calculates the relative position, *R*, between a rover and fixed base station to centimeter accuracy
- Integer ambiguity (IA), *N*, must be calculated
  - Many published algorithms available
  - Can take 20 minutes
  - New techniques utilizing L1 and L2 (wide laning) are nearly instantaneous

$$\lambda = \frac{c}{f} = 19\text{ cm for L1}$$

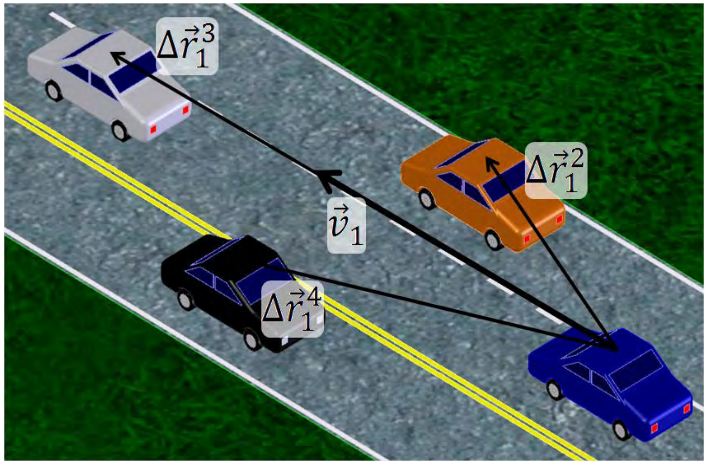
$$R = \left(N + \frac{\theta_2 - \theta_1}{360°} \right) \lambda$$

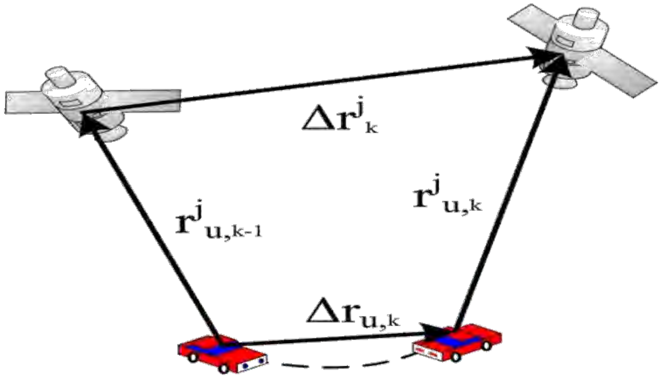
L1 $f = 1.5\ GHz$



Dynamic base RTK (DRTK)

- A *Real-Time Kinematic* (RTK) system is a form of *differential GPS* (DGPS)
  - A roving receiver in close proximity to a static base receiver (<50 km) differences measurements from both to cancel out common errors and estimate a high accuracy *relative position vector* (RPV)
  - The RPV is added to the known location of the base to produce a high accuracy global position
- *Dynamic base RTK* (DRTK) removes the constraint of having a static base station
  - An accurate RPV can be obtained, but global position accuracy is lost





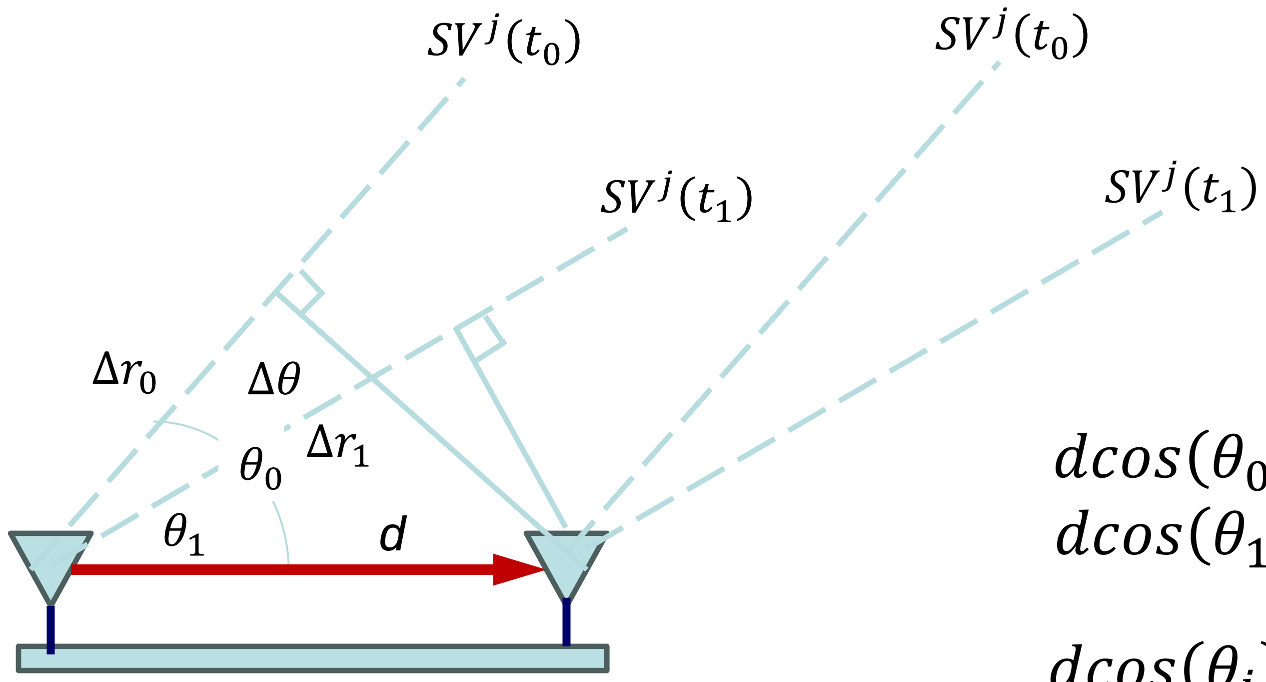
Integer Ambiguity

- The number of whole carrier cycles between a receiver and satellite
  - Carrier based DGPS technique utilizing the accuracy of a receiver's phase measurement
  - Estimates number of carrier cycles, *N*
    - Single frequency – Employs estimation scheme, can take up to 30 minutes
    - Dual frequency – A wide lane approach can limit the possibilities of *N*, drastically reducing search time
    - Triple frequency – *N* can be nearly instantaneously solved using a third frequency, such as L5, GLONASS, or GALILEO
  - Cycle slip is a sudden shift in the value of *N* when communication between satellite and receiver is compromised
    - Causes errors in RTK position fix

Differenced Code-Carrier Smoothing

- Since Ionosphere has been removed from single difference (SD), the SD pseudoranges and SD carrier measurements can be combined
  - No fear of code-carrier divergence (since Ionosphere has been removed)
    - Similar to dual frequency ionosphere-free pseudorange/carrier smoothing

Carrier Model



$$d\cos(\theta_0) = \lambda(\Delta\phi_0 + N_{ab})$$
$$d\cos(\theta_1) = \lambda(\Delta\phi_1 + N_{ab})$$

$$\frac{d\cos(\theta_i)}{\lambda} - N_{ab} = \Delta\phi_i$$

$$\phi_{ab}(t_0) = \phi_a(t_0) - \phi_a(t_0) = \Delta\phi_0 + N_{ab}$$

Carrier Model

- Can use the carrier model to attempt to estimate the integer ambiguity and distance between receivers:

$$\Delta\phi_i = \begin{bmatrix} \frac{\cos(\theta_i)}{\lambda} & -1 \end{bmatrix} \begin{bmatrix} d \\ N_i \end{bmatrix}$$

$$\Delta\Phi_i = \begin{bmatrix} \cos(\theta_i) & -\lambda \end{bmatrix} \begin{bmatrix} d \\ N_i \end{bmatrix}$$

- Therefore, assuming *d* and *N* are constant, we can use least squares:

$$\begin{bmatrix} \hat{d} \\ \hat{N}_i \end{bmatrix} = (H^T H)^{-1} \Delta\Phi_i$$

Carrier Model

- Can predict the accuracy of the estimates using our covariance matrix:

$$P = \sigma_{\Delta\Phi}^2 (H^T H)^{-1} = \begin{bmatrix} \sigma_{\hat{d}}^2 & \# \\ \# & \sigma_{\hat{N}_i}^2 \end{bmatrix}$$

- If we look at the integer ambiguity estimation error term in the covariance matrix:

$$\sigma_{\hat{N}_i}^2 = \sigma_{\Delta\Phi}^2 (IDOP)^2$$

- Can show that:

$$(IDOP)^2 = \frac{cos^2(\theta_1) + cos^2(\theta_0)}{(\cos(\theta_1) - \cos(\theta_0))^2}$$

Carrier Model

$$(IDOP)^2 = \frac{cos^2(\theta_1) + cos^2(\theta_0)}{(\cos(\theta_1) - \cos(\theta_0))^2}$$

- For small IDOP, need $cos(\theta_1) - cos(\theta_0)$ to be large
  - Must wait for SV motion
    - Baseline distance effects time to get sufficient Δθ
  - Alternatively, can artificially induce with “antenna swap”

General Integer Ambiguity (N) Estimation

- Could just compare pseudorange with carrier measurement
  - However as discussed earlier in the class you have code-carrier divergence
  - Could utilize dual frequency to get ionosphere free psuedorange and carrier measurement
    - Measurements are more noisy
  - With DGPS, can compare the single (or double) differenced psuedorange & carrier measurements:

$$\Delta\rho_{ab}^{j} = \vec{r}_{ab} + ct_{ab} + \epsilon_{\Delta\rho}$$

$$\Delta\Phi_{ab}^{j} = \vec{r}_{ab} + \lambda N_{ab}^{j} + ct_{ab} + \epsilon_{\Delta\rho}$$

Estimating Integer Ambiguity (N)

- Comparing the pseudorange & carrier measurements:

$$\Delta\rho_{ab}^{j} = \Delta r^{j} + ct_{ab} + \epsilon_{\Delta\rho}$$

$$\Delta\Phi_{ab}^{j} = \Delta r^{j} + \lambda N_{ab}^{j} + ct_{ab} + \epsilon_{\Delta\rho}$$

- Therefore:

$$\hat{N}_{ab}^{j} = \left[\frac{\Delta\Phi_{ab}^{j} - \Delta\rho_{ab}^{j}}{\lambda}\right]_{rounded}$$

- What is the problem?
  - The error (1σ) on *Δρ* is 0.5-1 meters
    - Corresponds to over 5 cycles of N (or 15 cycles at 3σ)
  - No Ionosphere divergence
    - Could average down over time

Estimating Integer Ambiguity (N)

- Could attempt to estimate N as part of the position solution

$$\Delta\rho_{ab}^{j} = \vec{r}_{ab} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{ab}^{j} = \vec{r}_{ab} + \lambda N_{ab}^{j} + ct_{ab} + \epsilon_{\Delta\rho}$$

- Have 5 unknowns ($\vec{r}_{ab}\,, t_{ab}\,, and\ N_{ab}^{j}$) and 2 equations
  - Each additional SV adds 2 equations and 1 more unknown (N)
    - With 4 common SVs (8 equations, 8 unknowns)
    - With 5 common SVs (10 equations, 9 unknowns)
    - With 6 common SVs (12 equations, 10 unknowns)

Estimating Integer Ambiguity (N)

- What about using additional frequencies?

$$\Delta\rho_{L1} = \vec{r} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{L1} = \vec{r} + \lambda N_{L1} + ct_{ab} + \epsilon_{\Delta\rho}$$

$$\Delta\rho_{L2} = \vec{r} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{L2} = \vec{r} + \lambda N_{L2} + ct_{ab} + \epsilon_{\Delta\rho}$$

- So we add 2 measurements and only one additional unknown
  - Obviously more equations with less unknowns helps in estimation
    - With 4 common SVs (16 equations and 12 unknowns)

Estimating Integer Ambiguity (N)

- What about "codeless" frequencies (i.e., legacy L2)

$$\Delta\rho_{L1} = \vec{r} + ct_{ab} + \epsilon_{\Delta\rho}$$
$$\Delta\Phi_{L1} = \vec{r} + \lambda N_{L1} + ct_{ab} + \epsilon_{\Delta\rho}$$

$$\Delta\Phi_{L2} = \vec{r} + \lambda N_{L2} + ct_{ab} + \epsilon_{\Delta\rho}$$

- Doesn't seem to be useful since adding one more unknown and only one more equation.
  - With 4 common SVs (12 equations and 12 unknowns)
  - However…

Frequency Combining

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- Recall what happens when you multiply two sine waves:

$$\sin(\omega_{L1}t) \times \sin(\omega_{L2}t) = \frac{1}{2}\sin([\omega_{L1} + \omega_{L2}]t) + \frac{1}{2}\sin([\omega_{L1} - \omega_{L2}]t)$$

- Therefore you get two new carrier frequencies (and two new resulting wavelengths)
  - The smaller wavelength is called the narrow-lane
  - The larger wavelength is called the wide-lane
- Interestingly, you don't actually have to physically multiply the signals
  - You get the same mathematical advantage simply by using both frequencies in the estimation process

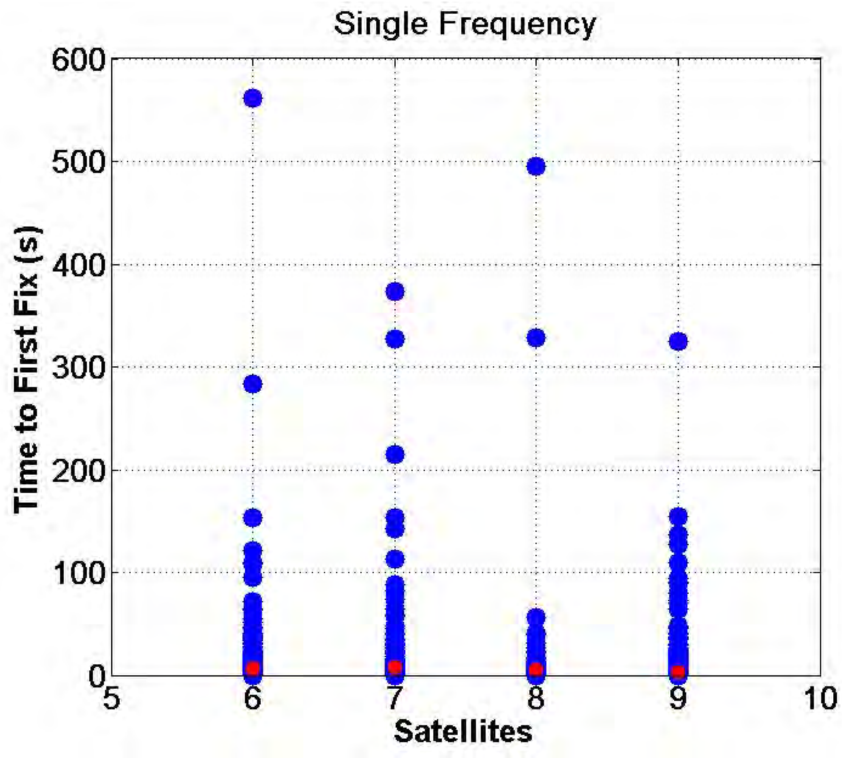
Multi-Frequency Benefits

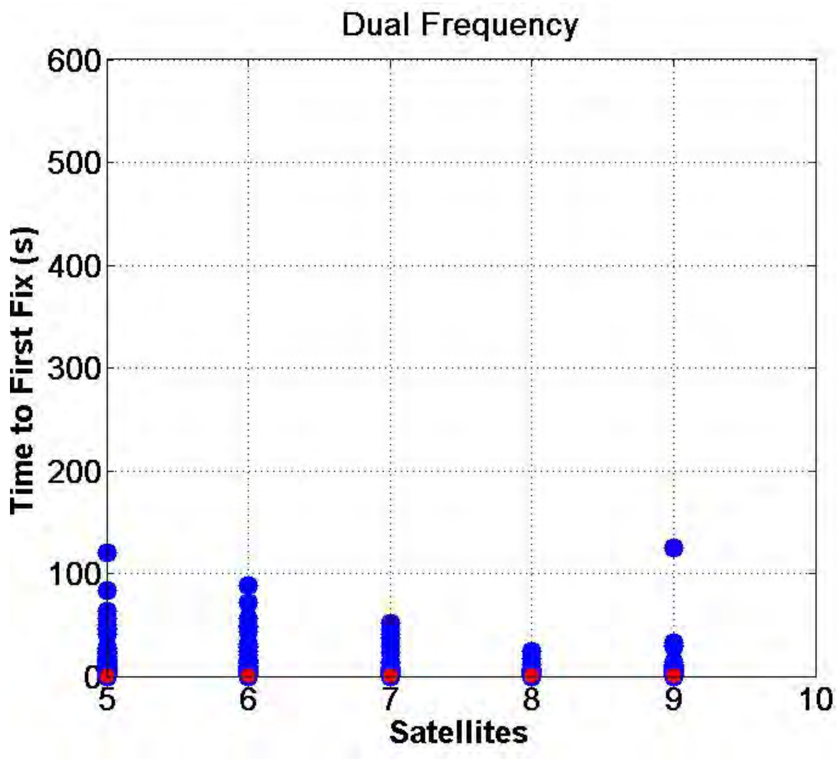
Combination	Wide Lane Wavelength (cm)	Narrow Lane Wavelength (cm)
L1-L2	86	10.70
L1-L5	75	10.89
L2-L5	586	12.47

- Dual frequency L1-L2 combination provides a wide lane near the accuracy of the single difference pseudorange (0.5-1 meter)
  - Just requires a few measurements to average to determine N
- Dual frequency L2-L5 provides instantaneous determination of N

RTK Integer Ambiguity Resolution

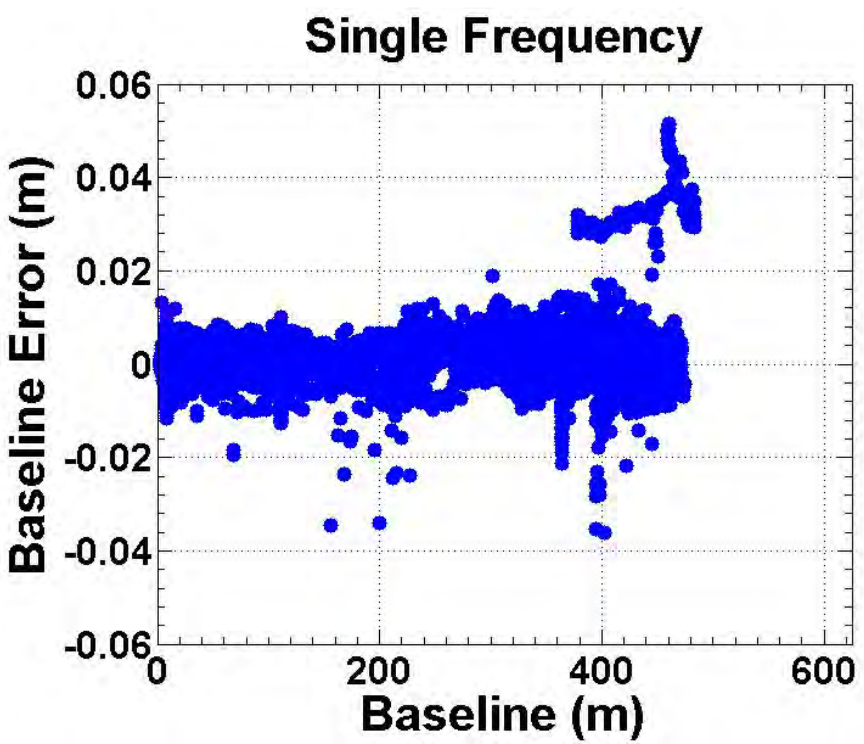
- Additional frequency drastically improves time to estimate N
  - Results shown are L1 vs L1-L2 codeless

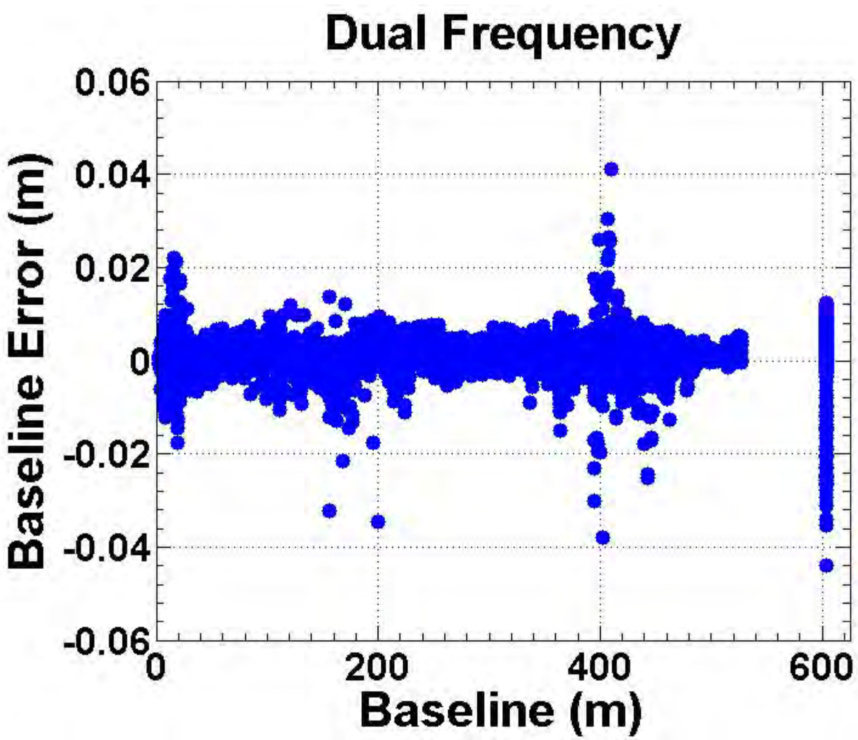




RTK Position Accuracy

- Additional frequency doesn't drastically improve accuracy (some averaging)





GPS Position Accuracy (1σ)

- Military Stand Alone (No SA) ~3m,
  - global coverage
- Civil Stand Alone (w/ SA) ~30m,
  - global coverage
- Code Phase Differential (DGPS) ~0.1m-1m
  - not all are global, but almost full US coverage
    - local reference station ~0.3m
    - Coast Guard differential corrections ~ 0.5m
    - WAAS ~1-3m
    - Nation Wide DGPS (NDGPS) ~ 1-3m
    - OmniStar VBS (~1m) & Omnistar HP (~10cm)
    - JohnDeere Starfire ~10cm
- Carrier Phase Differential (RTK) ~2cm,
  - local (~10km) coverage
    - High Accuracy (HA) NDGPS ~10 cm

Precise Point Positioning (PPP)

- Attempt to get carrier like position accuracy without base station
- Correct several source of error first:
  - Ideally, use dual frequency to remove ionosphere error
  - Use more precise ephemeris
    - https://igs.org/products/
  - Estimate troposphere error
    - Estimate as Tz*m
      - Where m is the elevation mapping

Precise Point Positioning (PPP)

- Generate Ionosphere Free Pseudorange:

$$\rho_{IF} = \frac{f_{L1}^2}{(f_{L1}^2 - f_{L2}^2)} \rho_{L1} - \frac{f_{L2}^2}{(f_{L1}^2 - f_{L2}^2)} \rho_{L2} = 2.546\rho_{L1} - 1.546\rho_{L2}$$

  - Noisier measurement:

$$\sigma_{\rho_{IF}} = \sqrt{(2.546)^2\sigma_{\rho_{L1}}^2 + (1.546)^2\sigma_{\rho_{L2}}^2} \approx 3\sigma_\rho$$

- Similarly, generate Ionosphere free carrier measurement:

$$\Phi_{IF} = \frac{f_{L1}^2}{(f_{L1}^2 - f_{L2}^2)} \Phi_{L1} - \frac{f_{L2}^2}{(f_{L1}^2 - f_{L2}^2)} \Phi_{L2} = 2.546\Phi_{L1} - 1.546\Phi_{L2}$$

Precise Point Positioning (PPP)

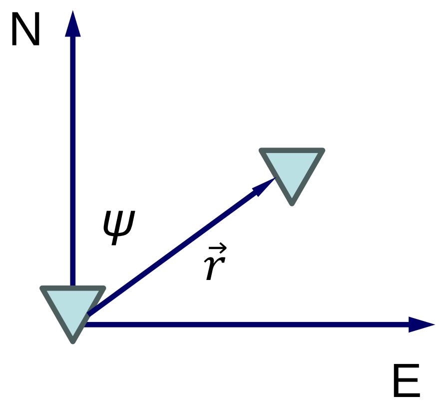
- Use the Ionosphere free pseudorange and carrier measurements:

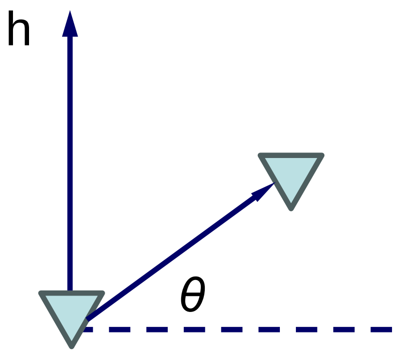
$$\rho_{IF} = \vec{r} + c\delta t_u + T_z \cdot m(el) + \epsilon_\rho$$

$$\Phi_{IF} = \vec{r} + c\delta t_u + T_z \cdot m(el) + \lambda N_{IF} + \epsilon_\Phi$$

- Must estimate position (x,y,z), clock bias, N_{IF}, and Tz
  - Need at least 5 SVs (10 equations and 10 unknowns)
  - If static, Tz changes a few cm/hour
  - Note: N_{IF} is no longer an integer
- Can achieve cm-level accuracy (~10 cm)
  - May require 15-30 minutes to converge in static mode
- Single frequency PPP is decimeter accurate (10-50 cm) with similar convergence rates

GPS RTK for Attitude





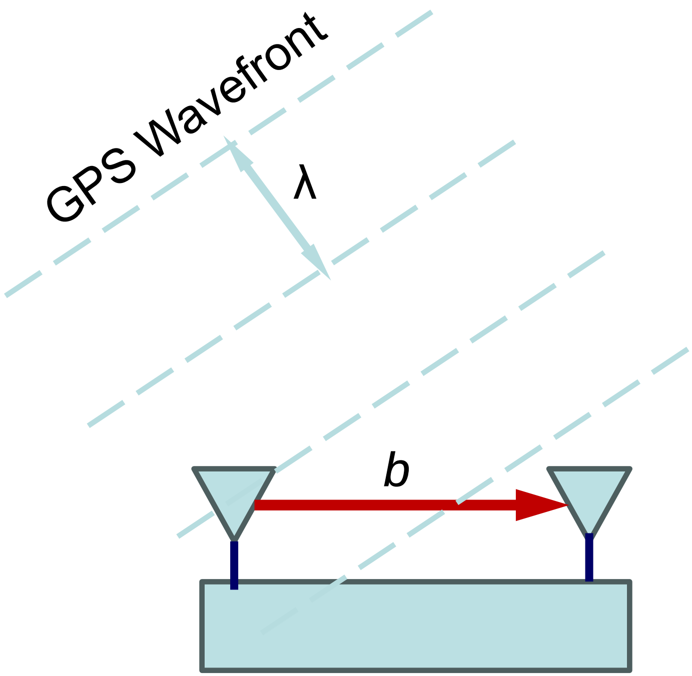
Convert $\vec{r}$ to ENU

$$\psi = tan^{-1}\left(\frac{\Delta E}{\Delta N}\right)$$

$$\theta = sin^{-1}\left(\frac{\Delta h}{L}\right)$$

- With 3 antennas (mounted rigidly on a body) can resolve roll, pitch, and yaw
- Accuracy is $\sigma \approx \frac{\sigma_{\Delta E}}{L}$
- Can use one common clock for all three receivers
  - Still have a line bias (however it is constant)

GPS Attitude



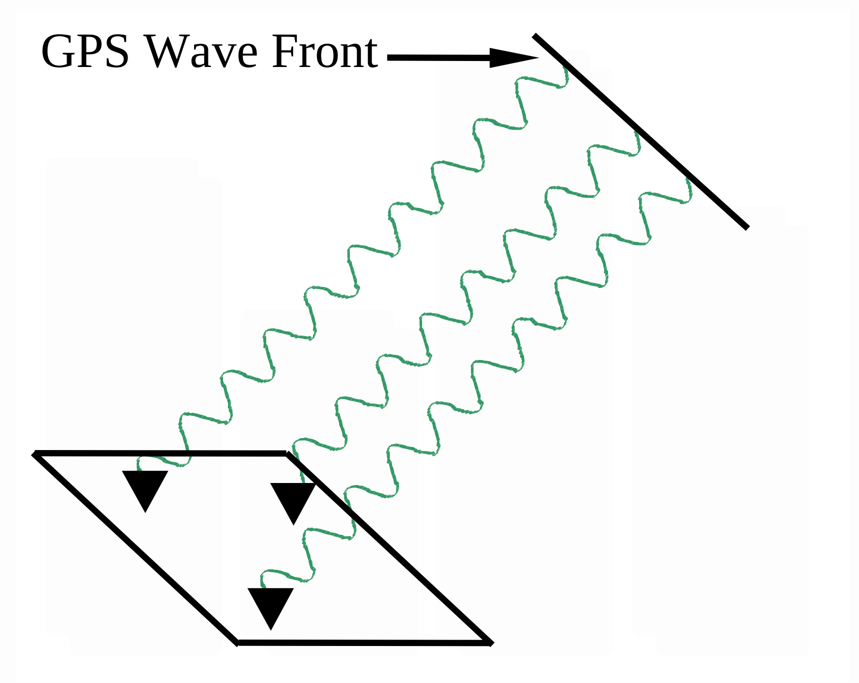
$$\Delta\Phi = b^T A\hat{s} + \lambda N + B + \nu$$

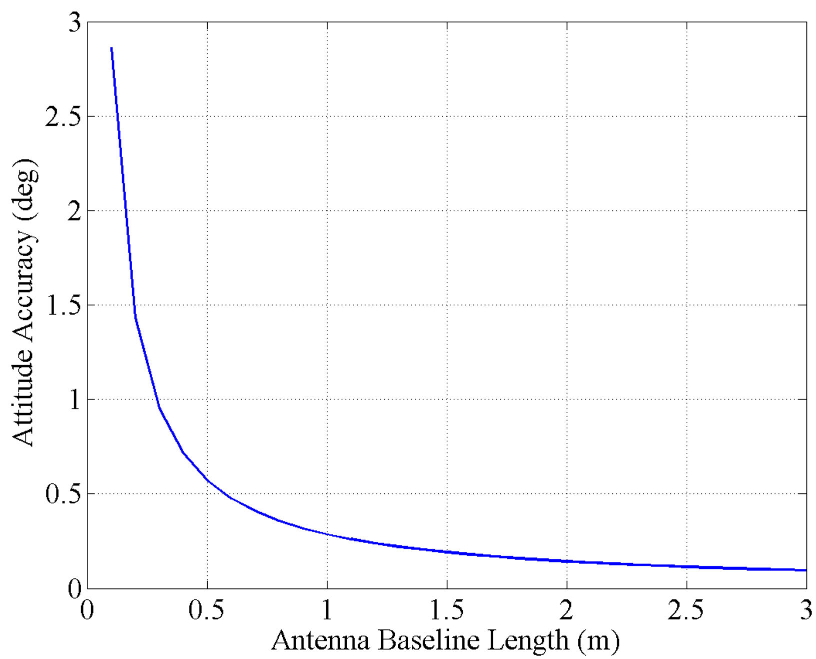
$B \Rightarrow$ Line Bias
$A \Rightarrow$ Direction Cosine Matrix from ENU to body frame
$N \Rightarrow$ Integer Ambiguity
$\nu \Rightarrow$ Noise/Error
$b \Rightarrow$ Baseline vector (in body frame)
$\hat{s} \Rightarrow$ Known unit vector in ENU frame

- Solve for A using cost function
  - Requires iterative search technique
  - Known as Wahba's Problem
  - Obtain roll, pitch, yaw (ϕ, θ, ψ) from A

GPS Attitude

No Reference Station Required
Accuracy Depends on Antenna Spacing





3 antennas ⇒ roll, pitch, yaw

accuracy = 0.1° (w/ 2m baseline)

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