5 Laminar Internal Flow

5.1 Governing Equations

These notes will cover the general topic of heat transfer to a laminar flow of fluid that is moving through an enclosed channel. This channel can take the form of a simple circular pipe, or more complicated geometries such as a rectangular duct or an annulus. It will be assumed that the flow has constant thermophysical properties (including density). We will first examine the case where the flow is fully developed. This condition implies that the flow and temperature fields retain no history of the inlet of the pipe. In regard to momentum conservation, FDF corresponds to a velocity profile that is independent of axial position \( x \) in the pipe. In the case of a circular pipe, \( u = u(r) \) where \( u \) is the axial component of velocity and \( r \) is the radial position. The momentum and continuity equations will show that there can only be an axial component to velocity (i.e., zero radial component), and that

\[
u(r) = 2u_m \left( 1 - \left( \frac{r}{R} \right)^2 \right)
\]

with the mean velocity given by

\[
u_m = \frac{1}{A} \int_A u \, dA = \frac{2}{R} \int_0^R u \, r \, dr
\]

The mean velocity provides a characteristic velocity based on the mass flow rate;

\[\dot{m} = \rho u_m A\]

where \( A \) is the cross sectional area of the pipe.

The thermally FDF condition implies that the dimensionless temperature profile is independent of axial position. The dimensionless temperature \( T \) is defined by

\[T = \frac{T - T_s}{T_m - T_s} = \text{func}(r), \quad \text{TFDF}\]

in which \( T_s \) is the surface temperature (which can be a function of position \( x \)) and \( T_m \) is the mean temperature, defined by

\[T_m = \frac{1}{u_m A} \int_A u \, T \, dA\]

This definition is consistent with a global 1st law statement, in that

\[\dot{m} C_P (T_{m,2} - T_{m,1}) = \dot{Q}_{1-2}\]

with \( C_P \) being the specific heat of the fluid. A differential form of this law is

\[\dot{m} C_P \frac{dT_m}{dx} = q''_s \frac{dA_s}{dx} = q''_s \mathcal{P}\]

in which \( q''_s \) is the surface heat flux, which may be a function of \( x \), and \( A_s \) and \( \mathcal{P} \) are the pipe surface area and perimeter. Note that integration of Eq. (7) over the length of the pipe results in Eq. (6).

The thermally FDF condition, in Eq. (4), also states that the heat transfer coefficient \( h \) is constant. The convection law for internal flow is defined by

\[q''_s = h(T_s - T_m) = k \frac{\partial T}{\partial r} \bigg|_R = -k(T_s - T_m) \frac{dT}{d\eta} \bigg|_1\]

with \( \eta = r/R \) being the dimensionless \( r \) coordinate. There is no negative sign in Fourier’s law, because the radial coordinate points toward the surface. Rearranging,

\[Nu_D = \frac{hD}{k} = -2 \frac{dT}{d\eta} \bigg|_1 = \text{constant}\]

The governing DE for the temperature distribution in the fluid is

\[\rho C_P u \frac{\partial T}{\partial x} = k \nabla^2 T = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial x^2} \right)\]
with boundary conditions (for the circular pipe case)

\[ \frac{\partial T}{\partial r} \bigg|_0 = 0, \quad T(r = R) = T_s \quad \text{or} \quad -k \frac{\partial T}{\partial r} \bigg|_R = q''_s \] (11)

The specific form of the wall BC would obviously depend on the given conditions, i.e., specified temperature, heat flux, or something more complicated. A convection-type BC is not an option here, as we are trying to determine the heat transfer coefficient \( h \) from a detailed formulation of convective–diffusive energy transport. No BCs are given for the axial direction (we would need two, typically an inlet and an outlet condition, to close the problem) because, as will be developed below, such will not be needed for the FDF case.

### 5.2 Fully Developed Flow

The second term on the right hand side of Eq. (10), which accounts for axial diffusion of heat, can usually be neglected in comparison to the first for the FDF condition. Nevertheless, Eq. (10) remains a partial DE (2nd order in \( r \), 1st in \( x \)), and it would appear that techniques used to solve PDEs must be employed. This, however, will not be the case per the FDF condition: an ODE will emerge once the equation is recast with \( T \) as the dependent variable. Doing this, however, requires addressing the convection term on the left hand side.

Using Eq. (4), it follows that

\[ \frac{\partial T}{\partial x} = \frac{dT_s}{dx} (1 - T) + \frac{dT_m}{dx} T \] (12)

Rather than substituting this formula into Eq. (10), it is simpler to first consider the form of the boundary condition at the wall. Two important cases are constant heat flux and constant \( T_s \).

#### 5.2.1 Constant heat flux

From Eqs. (8) and (12), the \( q''_s \) = constant case results in

\[ \frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} \] (13)

The differential energy balance in Eq. (7) can be used to relate the axial temperature derivative to \( q''_s \), and Eq. (8) can then be used to replace \( q''_s \) with the convection law. Likewise, the radial conduction (or diffusion) term on the right can be made dimensionless via introduction of \( \overline{T} \) and \( \overline{r} \). Even though both \( T_s \) and \( T_m \) will be functions of \( x \) for this case, they can be brought inside the radial derivatives (either multiplicatively or additively) because they are not functions of \( r \). After rearranging terms and using the circular pipe configuration, the result is

\[ -\overline{\pi} Nu_D \frac{1}{\overline{r}} \frac{d}{d\overline{r}} \left( \overline{r} \frac{dT}{d\overline{r}} \right) \] (14)

in which \( \overline{u} = u/u_m \) is the dimensionless velocity profile. The DE is ordinary, as opposed to partial, because of \( \overline{T} \) being a function solely of \( \overline{r} \). The boundary conditions are

\[ \frac{\partial T}{\partial \overline{r}} \bigg|_0 = 0, \quad \overline{T}(1) = 0 \] (15)

Equation (14) is an inhomogeneous ODE, with two homogeneous BCs in Eq. (15). The solution, which can be obtained by direct integration, is

\[ \overline{T} = \frac{Nu_D}{8} (3 - 4\overline{r}^2 + \overline{r}^4) \] (16)

The Nusselt number – which is currently unknown and is sought from the analysis – is obtained from the mean temperature condition. On a dimensionless basis, the mean temperature is

\[ \overline{T}_m = 2 \int_0^1 \overline{T} \overline{r} d\overline{r} = 1 \] (17)

and this results in

\[ Nu_D = \frac{48}{11} \approx 4.364, \quad \text{constant } q''_s \] (18)
5.2.2 Constant surface temperature

From Eq. (12), the $T_s = \text{constant}$ case results in

$$\frac{\partial T}{\partial x} = \frac{dT_m}{dx} \bar{T}$$

(19)

By comparison with Eq. (13), the only change from the $q_s'' = \text{constant}$ formulation is that the convection term will now contain $\bar{T}$, i.e.,

$$-\pi Nu_D \bar{T} = \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \left( \bar{r} \frac{d\bar{T}}{d\bar{r}} \right)$$

(20)

The boundary conditions are the same as before, and the mean temperature condition of Eq. (17) holds as well.

Equation (20) is an eigenvalue problem, the eigenvalue being the Nusselt number $Nu_D$. Observe that both the DE and the BCs are homogeneous\(^1\). Therefore, there can only exist non–trivial (i.e., non–zero) solutions to this problem for a unique value (or values) of the eigenvalue $Nu_D$. A simple example of an eigenvalue problem is the homogeneous boundary value problem

$$f''(x) + c^2 f(x) = 0, \quad f'(x = 0) = 0, \quad f(x = 1) = 0$$

The solution to this problem is

$$f = \cos(cx)$$

and the solution also has an infinite number of eigenvalues $c$:

$$c = \pi/2, \ 3\pi/2, \ 5\pi/2, \ldots$$

The problem we face in Eq. (20) is much the same. Unlike the simple example given above, however, there is no general solution to the ODE in Eq. (20) – at least not in terms of commonly–known mathematical functions like sin and cos. It turns out that a general solution to Eq. (20), in the form of ordinary Bessel functions, can be found for the special case of $\pi = 1$ (a case known as *slug flow*). Our interest, however, is for the laminar, parabolic flow case.

An iterative strategy can be used to determine $Nu_D$ for the constant temperature case. The approach is to write Eq. (20) as

$$-\pi Nu_{Dn} T_n = \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \left( \bar{r} \frac{dT_n}{d\bar{r}} \right)$$

(21)

where $T_n$ refers to the $n^{th}$ iterated solution to Eq. (21). The BCs on $T_n$ are the same as before. Note that $T_n(\bar{r})$ will be a function of $T_{n-1}$ and $Nu_{Dn}$. In addition, each iterate $T_n$ must satisfy the mean condition

$$2 \int_0^1 \bar{r} T_n \bar{r} d\bar{r} = 1$$

(22)

and this provides a relation to determine $Nu_{Dn}$. For a sufficiently large number of iterations $n$, the procedure will give $Nu_{Dn-1} \approx Nu_{Dn} = Nu_D$, i.e., the iterated Nusselt numbers will converge to a constant.

A starting point in the process is to let $T_0 = 1$. You should be able to show, using the previous section results, that the first iteration gives $Nu_{D1} = 48/11$, i.e., the constant $q_s''$ result. Each iterated solution $T_n$ can be obtained by direct integration of Eq. (21). For the given $T_0$, each solution $T_n$ will be a polynomial in $\bar{r}$ of degree $4n$ – and you can see that the algebra can quickly get complicated. It helps to use a symbolic package such as *Mathematica* or *Maple* to solve this problem.

The following illustrates how the problem would be solved in *mathematica*. The first two lines set the velocity profile and the $T_0 = 1$ condition:

```plaintext
u[2,] := 2(1 - r^-2); tnm1[2,] := 1
```

and the next set of instructions find the $T_n = tnm[2]r$ solution and $Nu_{Dn}$ value using the current value of $T_{n-1} = tnm1[2]$. Note that the last line updates the iteration, i.e., $tm1[2]$ is replace $tm[2]$.

\(^1\)The definition of a homogeneous equation $F(y) = 0$ is that $F(cy) = 0$ for an arbitrary constant $c$. 

\[
de = -u[r] \text{nud} tnm1[r] == 1/r D[r \text{D}[t \text{n}[r], r], r];
bc1 = t \text{n}[0]'[0] == 0;
bc2 = t \text{n}[1]'[1] == 0;
soln = DSolve\{\{de, bc1, bc2\}, t \text{n}[r], r\}[[1, 1]]
meant = 2 \text{Integrate}\{u[r] t \text{n}[r] r /. soln, \{r, 0, 1\}\};
nudsoln = \text{Solve}\{\text{meant} == 1, \text{nud}\}[[1, 1]]
tnm1[r_] := \text{Evaluate}\{t \text{n}[r] /. soln /. nudsoln\}
\]

Results are as follows

\[n=1\]
\[\text{tn}[r] \rightarrow (1/8)*(3*\text{nud} - 4*\text{nud}*r^2 + \text{nud}*r^4)\]
\[\text{nud} \rightarrow 4.36364\]

\[n=2\]
\[\text{tn}[r] \rightarrow (1/528)*(251*\text{nud} - 432*\text{nud}*r^2 + 252*\text{nud}*r^4 - 80*\text{nud}*r^6 + 9*\text{nud}*r^8)\]
\[\text{nud} \rightarrow 3.72881\]

\[n=3\]
\[\text{tn}[r] \rightarrow (1/28320)*(13881*\text{nud} - 25100*\text{nud}*r^2 + 17075*\text{nud}*r^4 - 7600*\text{nud}*r^6 + 2075*\text{nud}*r^8 - 356*\text{nud}*r^{10} + 25*\text{nud}*r^{12})\]
\[\text{nud} \rightarrow 3.66723\]

\[n=4\]
\[\text{nud} \rightarrow 3.65844\]

\[n=5\]
\[\text{nud} \rightarrow 3.65706\]

The limiting value is \[\text{Nu}_D = 3.65679\]

**Exercises**

1. Consider laminar flow between two parallel plates. The flow is incompressible, has constant properties, and is fully developed.
   (a) Derive the velocity profile and the mean velocity.
   (b) Derive the Nusselt number for the constant surface heat flux case (both surfaces are heated)
   (c) Derive the Nusselt number for the case in which one surface is insulated, and the other is maintained at a constant temperature. Your should perform at least two iterations for the temperature profile.

2. A counterflow heat exchanger consists of a simple pair of coannular tubes (i.e., one tube running inside another). Say that the outer tube contains the hot fluid, and the inner the cold. Both fluids are otherwise identical (the same substance) and have the same mass flow rate. Both flows are laminar and fully developed. The outer wall of the pipe is insulated, so that heat is only exchanged between the two fluids.
   (a) Derive the velocity profile and mean velocity in the outer (annular) pipe.
   (b) Show that, for this particular situation, the heat transfer problem for both the cold and hot fluids is equivalent to the constant surface heat flux model.
   (c) Derive formulas for the Nusselt numbers for the inner and outer tubes.
5.3 Thermally developing flow

A thermally developing flow occurs when the thermal boundary conditions change from one state to another at some point \( x = 0 \) along the pipe. We will assume the flow is hydrodynamically developed throughout the pipe. For example, the wall temperature \( T_s \) could jump from one value to another at \( x = 0 \). At a sufficient distance downstream from this point the temperature profile will attain the fully–developed limit given by Eq. (4). Of interest here is prediction of the heat transfer that occurs up to the fully–developed point.

For constant thermal boundary conditions (i.e., constant wall temperature or heat flux), the local heat transfer coefficient \( h_x \) will be at a maximum value at the starting point, and will monotonically decrease to the fully developed limit. The length required for a flow to become fully developed can be estimated simply via dimensional analysis. Neglecting axial diffusion – which is appropriate for all except very slow–moving flow – the energy equation is

\[
\rho C_p u_m \pi \frac{\partial T}{\partial x} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\]  

(23)

with \( \pi(r) = u(r)/u_m \). Now let

\[
\theta = \frac{T - T_s}{T_e - T_s}, \quad \tau = \frac{r}{R}, \quad \bar{x} = \frac{2x}{R P e_D} = \frac{k x}{\rho C_p u_m R^2}
\]

(24)

in which \( T_e \) is the inlet temperature at \( x = 0 \) and

\[
P e_D = \frac{u_m D}{\alpha} = R e_D Pr
\]

(25)

is the Peclet number of the flow. The Peclet number represents a ratio of characteristic times for diffusion and convection of heat. The typical conditions of \( P e_D \gg 1 \) allow us to neglect the axial diffusion term in the energy equation. The dimensionless energy equation becomes

\[
\frac{\pi}{\tau} \frac{\partial \theta}{\partial \bar{x}} = \frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial \theta}{\partial \tau} \right)
\]

(26)

with boundary conditions of

\[
\theta(x = 0) = 1, \quad \frac{\partial \theta}{\partial \tau}\bigg|_{\tau = 0} = 0, \quad \theta(\tau = 1) = 0
\]

(27)

The above DE and BCs represent a well–posed problem. There are no free parameters in the problem, and the solution \( \theta \) must be a function solely of \( \tau \) and \( \bar{x} \). The local Nusselt number will be obtained from

\[
Nu_D = \frac{h_x D}{k} = -2 \frac{\partial \bar{T}}{\partial \bar{x}} \bigg|_{\bar{x} = 1} = -2 \frac{T_e - T_s}{T_{m} - T_s} \frac{\partial \theta}{\partial \tau}\bigg|_{\tau = 1} = -2 \frac{\theta_m}{\partial \tau}\bigg|_{\tau = 1}
\]

(28)

with

\[
\theta_m(\tau) = 2 \int_0^1 \theta(\tau, \pi) \pi \pi d\pi
\]

(29)

Regardless of the actual thermophysical properties of the fluid, the length required for the flow to become FD – for which \( Nu_D \) will be within, say, 0.99 of the FD \( Nu_D \) – will correspond to some constant value \( \bar{x} \). Therefore,

\[
\bar{x}_{fd} = C_1 \quad \rightarrow \quad x_{fd} = C_1 R P e_D
\]

(30)

A solution to Eq. (20) will be needed to evaluate the constant \( C_1 \), yet dimensional considerations dictate that the entrance length must be proportional solely on the pipe radius and \( P e_D \).

A dimensionless global energy conservation statement can be developed by multiplying Eq. (26) by \( 2\pi \pi d\pi \), integrating from 0 to 1, using the mean temperature definition in Eq. (29), and combing the result into Eq. (28):

\[
\frac{d\theta_m}{d\bar{x}} = -Nu_D \theta_m
\]

(31)

and using \( \theta_m(0) = 1 \)

\[
\theta_m = \exp \left[ - \int_0^{\bar{x}} Nu_D \bar{x} \right]
\]

(32)
Now define an averaged Nusselt number at point $x$, $\overline{N_{uD_x}}$, as

$$\overline{N_{uD_x}} = \frac{1}{x} \int_0^x N_{uD_x} \, dx$$

(33)

so that

$$\theta_m = \exp \left[ -\overline{N_{uD_x}} \, x \right] = \exp \left[ -\frac{2\overline{\theta} \, x}{\rho u_m C_P R} \right] = \exp \left[ -\frac{\overline{h_x A_s}}{\dot{m} C_P} \right]$$

(34)

This should be a familiar heat transfer statement.

### 5.3.1 Separation of variables solution

The remaining task is to actually solve Eq. (26). This can be accomplished via separation of variables methods, although doing so leads into some mathematically dense territory. SOV first begins by assuming a separable form to the solution,

$$\theta = f(x) \phi(r)$$

(35)

I have dropped the overline notation on the variables – yet it is implied that they are dimensionless. Replacing this into the DE and separating the variables;

$$\frac{f'}{f} = \frac{1}{r \pi \phi} (r\phi')'$$

(36)

The left hand side is a function of $x$ and the right hand side of $r$. The only way the equation can hold is if both sides equal a constant,

$$\frac{f'}{f} = \frac{1}{r \pi \phi} (r\phi')' = -\lambda^2$$

(37)

The reason for the negative sign on the constant $\lambda^2$ will soon become apparent, and the reason for the square is to force $\lambda^2$ to be positive (it is assumed that $\lambda$ is real).

Two ODEs emerge:

$$f' = -\lambda^2 f$$

$$r\phi' = 0$$

(38)

The solution for $f$ is a decaying exponential – which makes sense as $\theta \to 0$ for $x \to \infty$. There is no solution to Eq. (39) in terms of well–known functions; had $\pi = 1$ (so–called slug flow) the solution would involve the ordinary Bessel function $J_0(\lambda r)$. This lack of an accessible solution provides a great opportunity to outline a procedure for solving homogeneous and linear DEs (of which Eq. (39) belongs) from scratch, i.e., the solution will be developed from the ground up, rather than using a cookbook to select the special function which satisfies the DE.

The approach is to formulate the solution as a power series in $r$;

$$\phi(r) = \sum_{n=0}^{\infty} a_n \, r^n$$

(40)

Negative powers are excluded because the solution cannot blow up at $r = 0$. Use this form in Eq. (39), and substitute the parabolic formula for $\pi$;

$$\sum_{n=0}^{\infty} a_n \left( a_n^{n+1} + 2\lambda^2 (r^{n+1} - r^{n+3}) \right) = 0$$

(41)

Now collect terms with like powers of $r$;

$$\sum_{n=0}^{\infty} r^n \left( (n + 1)^2 a_{n+1} + 2\lambda^2 (a_{n-1} - a_{n-3}) \right) = 0$$

(42)

It is implied in the above that $a_n = 0$ for $n < 0$. This relation must hold for all $r$, so we can conclude that each term in the series must be zero. This gives a recurrence relation for the expansion coefficients $a_n$;

$$(n + 2)^2 a_{n+2} = -2\lambda^2 (a_n - a_{n-2}), \quad n = 0, 1, 2, \ldots$$

(43)
Since \(a_{-2} = 0\), then
\[
a_2 = -\frac{\lambda^2}{2} a_0
\]
and
\[
a_4 = \frac{2\lambda^2}{4^2} (a_2 - a_0) = -\frac{2\lambda^2}{4^2} (-\frac{\lambda^2}{2} - 1) a_0
\]
(44)
The even-numbered coefficients will therefore all be proportional to \(a_0\).

The same procedure can be applied for the odd powers of \(n\), and we would find that \(a_3, a_5, \ldots\) are all proportional to \(a_1\). However, we can exclude the odd-powers from our series because, according to the required boundary condition on the solution, the derivative of \(\phi\) must vanish at \(r = 0\). According to Eq. (40), \(\phi'(0) = a_1\), and consequently \(a_1 = 0\).

For computational purposes we can arbitrarily set \(a_0 = 1\), for which \(a_2 = -\lambda^2/2\). The higher-order coefficients are then calculated via
\[
a_{n+2} = \frac{2\lambda^2}{(n+2)^2} (a_n - a_{n-2}), \quad n = 2, 4, 6, \ldots
\]
(45)
for a sufficiently large number of terms. The \(a_n\) will generally grow in magnitude until \(n \approx \lambda\), and will decrease after this point.

This algorithm will fail when \(\lambda^2\) gets large due to numerical lack-of-precision errors. The individual terms in the series for \(n \approx \lambda\) will get huge (with alternating sign), yet the sum will be of order unity. The process of computing a small number on a computer via the subtraction of two large numbers will result in a loss of precision in the number: to see how this works, try calculating \(10^n - (10^n - 1)\), and for sufficiently large \(n\) you will no longer get the exact (and trivial) result of 1.

With the above caution in mind, the function \(\phi\) can be calculated as
\[
\phi(r; \lambda) = 1 + \sum_{n=1}^{\infty} a_{2n} r^{2n}
\]
(46)
The argument \(\lambda\) is included to emphasize that \(\phi\) will be a function of this constant.

The value of \(\lambda\) remains to be decided. The solution must satisfy the wall condition of \(\theta(r = 1) = 0\), which implies that \(\phi(r = 1; \lambda) = 0\). Accordingly, \(\lambda\) must be a root to the equation
\[
\phi(1; \lambda) = 0
\]
(47)
It turns out that there are an infinite number of roots to this equation. The function \(\phi(1; \lambda)\), defined by Eq. (46), will have an oscillatory behavior with increasing \(\lambda\). This behavior is illustrated in the plot shown below. Note that strange behavior occurs in \(\phi(1; \lambda)\) for \(\lambda \approx 25\); this is a result of the numerical LOP errors discussed above. Such functions are referred to as transcendental functions. A common such function is \(\cos(\lambda r)\), and the condition \(\cos(\lambda) = 0\) implies that \(\lambda = \pi/2, 3\pi/2, \ldots\). Unfortunately, no explicit formula exists for the roots \(\lambda\) of Eq. (47), and they must be obtained numerically.

Denote \(\lambda_1\) as the first root (beginning from zero) to Eq. (47), \(\lambda_2\) the next root, and so on. These quantities are denoted as the eigenvalues to the eigencondition of Eq. (47). The function \(\phi(r; \lambda)\), evaluated with a given eigenvalue \(\lambda_n\), will be referred to as the eigenfunction \(\phi_n(r)\). Each eigenfunction can have a unique contribution to the solution for \(\theta\) – there is no rationale at this point for excluding any particular one – so
we must include all possible solutions in developing a sufficiently general solution to Eq. (26). The function \( f(x) \), given as the solution to Eq. (38), will also depend on the eigenvalue \( \lambda_n \), so that
\[
f_n(x) = \exp(-\lambda_n^2 x)
\] (48)
The general solution to Eq. (26) therefore appears as an infinite series;
\[
\theta(r, x) = \sum_{n=1}^{\infty} A_n \phi_n(r) \exp(-\lambda_n^2 x)
\] (49)
in which the expansion coefficients \( A_n \) are yet to be determined. You should note that the index \( n \) in Eq. (49) is simply a dummy variable and is in no way connected to the index \( n \) (another dummy variable) in Eq. (46). If this is confusing, replace \( n \) in Eq. (46) with some other symbol, i.e., \( l \).
The expansion coefficients are obtained from the initial condition at \( x = 0 \), for which
\[
\theta(r, 0) = 1 = \sum_{n=1}^{\infty} A_n \phi_n(r)
\] (50)
The eigenfunction has the useful property of orthogonality which allows the series in the above formula to be "inverted", so that an explicit formula can be derived for each \( A_n \). This property, for the specific form of the DE in Eq. (39), states that
\[
\int_0^1 \phi_n \phi_m \, \overline{r} \, dr = \begin{cases} 0, & n \neq m \\ \text{not } 0, & n = m \\ \end{cases}
\] (51)
To prove this, the integration is performed by substituting from the DE in Eq. (39) and using integration by parts:
\[
\int_0^1 \phi_n \phi_m \, \overline{r} \, dr = -\frac{1}{\lambda_n^2} \int_0^1 (r \phi'_n) \phi_m \, dr
\]
\[
= -\frac{1}{\lambda_n^2} \left( [r \phi'_n \phi_m - r \phi_n \phi'_m]_0^1 + \int_0^1 \phi_n (r \phi'_m) \, dr \right)
\]
\[
= \frac{\lambda_n^2}{\lambda_m^2} \int_0^1 \phi_n \phi_m \, \overline{r} \, dr
\] (52)
The boundary terms (represented by the terms in square brackets on the second line) are identically zero via the boundary conditions satisfied by \( \phi \), i.e., \( \phi'(0) = \phi(1) = 0 \). It can therefore be concluded that
\[
\left( 1 - \frac{\lambda_m^2}{\lambda_n^2} \right) \int_0^1 \phi_n \phi_m \, \overline{r} \, dr = 0
\] (53)
for all \( n \) and \( m \). When \( n \neq m \), for which the term in parenthesis is not zero, the integral must evaluate to zero. And for \( n = m \) the result does not tell us anything about the value of the integral – although we can conclude that it must be positive since \( \phi_n^2 \, r \, \overline{u} \geq 0 \) on \( 0 \leq r \leq 1 \).
The derivation performed above indicates that the orthogonality property of \( \phi_n \) is a consequence solely of the DE, Eq. (39), and the fact that \( \phi_n \) satisfies homogeneous boundary conditions. Differential equations which have this property are known as Sturm–Liouville systems.
To evaluate the coefficients in Eq. (50), multiply each side by \( \phi_m \overline{r} \, dr \) and integrate from 0 to 1. Each term in the series disappears, by virtue of Eq. (53), except the one with \( m = n \). The result is
\[
A_m = \frac{\int_0^1 \phi_m \overline{r} \, dr}{\int_0^1 \phi_m^2 \overline{r} \, dr}
\] (54)
The integral in the numerator can be evaluated using the DE, Eq. (39),
\[
\int_0^1 \phi_m \overline{r} \, dr = -\frac{1}{\lambda_m^2} \int_0^1 (r \phi'_m)' \, dr = -\frac{\phi'_m(1)}{\lambda_m^2}
\] (55)
I am not aware of some analytical trick to evaluate the integral in the denominator. It could be done numerically, or via brute–force via the series formula for \( \phi_n \):

\[
\int_0^1 \phi_n^2 \pi r dr = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{2k} a_{2l} g(2(k + l))
\]

in which

\[
g(n) = \int_0^1 r^n \pi(r) r dr = \frac{4}{8 + 6n + n^2}
\]

The solution for the developing flow problem, given by Eq. \(49\) is now complete. The mean dimensionless temperature is

\[
\theta_m(x) = 2 \int_0^1 \theta(x, r) \pi r dr = 2 \sum_{n=1} A_n \exp(-\lambda_n^2 x) \int_0^1 \phi_n(r) \pi r dr = -2 \sum_{n=1} \frac{A_n \phi'_n(1)}{\lambda_n^2} \exp(-\lambda_n^2 x)
\]

The local Nusselt number, from Eq. \(28\), is

\[
Nu_{Dx} = \left. \frac{-2 \theta_m}{\theta_m} \frac{\partial \theta}{\partial r} \right|_1 = -2 \frac{\theta_m}{\theta_m} \sum_{n=1} A_n \phi'_n(1) \exp(-\lambda_n^2 x)
\]

and the mean Nusselt number, from Eq. \(34\), is simply

\[
\overline{Nu_{Dx}} = \frac{-1}{x} \ln[\theta_m(x)]
\]

The number of terms in the series solution, Eq. \(49\), needed for the series to converge will decrease with increasing \(x\). For \(x \approx 1\) only the \(n = 1\) term is required for acceptable accuracy. Using the single–term limits in Eq. \(58\) and \(59\) results in

\[
Nu_{Dx} \left|_{FD} \rightarrow \lambda_1^2 = 3.66
\]

The first term in the series therefore corresponds to the fully–developed, constant wall temperature solution. Recall that the FD solution was derived in the first section via an iterative substitution procedure; it could have been also been derived using the power series method that was applied to Eq. \(39\).

Some illustrative results, calculated using Mathematica, are shown below.
The plot on the top shows the dimensionless $\theta/\theta_m$ profiles as a function of $r$ for $\pi = 0.01, 0.02, 0.05, 0.1,$ and 1. My code could only compute 8 terms in the series before the power-series algorithm in Eq. (41) failed, and the results for $\pi = 0.01$ show some noise which is due to an insufficient number of terms in the series. Note, however, that the temperature does go from a uniform, unit-value profile near the entrance, to the fully-developed profile.

The bottom plot shows the local (blue) and averaged (red) Nusselt numbers as a function of $x$. The local goes to the FD limit for $x \approx 0.2$, yet the average value takes a considerably longer length of pipe before it attains the FD limit.