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Solving the Colebrook Equation for Friction Factors

Introduction:

In 1944 Lewis F. Moody, Professor, Hydraulic Engineering, Princeton University, published "Friction Factors for Pipe Flow"¹. The work of Moody, and the Moody Diagram on page 672 of the published transactions, has become the basis for many of the calculations on friction loss in pipes, ductwork and flues. While there are modified versions of the original Moody Diagram, I will strive to use the original diagram as the basis for terminology used here.

Moody references the work of C.F. Colebrook and C.M. White, amongst others, in developing his Moody Diagram. The Moody Diagram can be used as a graphical solution of the Colebrook Equation. There are tools available today that allow solution of the Colebrook Equation, in both its Implicit forms and Explicit forms, without using the graphical approach. These are much more useful when working with electronic spreadsheets as will be done in this series.

The series will examine:

Implicit Forms of Colebrook:

We will look at the three common forms of the Colebrook Equation. The differences between these three equations will be examined and the deviations in the results that they produce will be explored.

User Defined Functions (UDF) for the Implicit Forms of Colebrook:

We will look at UDFs that solve the three Implicit forms of Colebrook. The functions are written in the Visual Basic Editor which is part of Excel spreadsheets. The accuracy of the UDFs will be compared to those obtained by iteration.

Explicit Forms of Colebrook:

In this section we will be looking at four forms of the Colebrook Equations that I am familiar with. I would also like to solicit the readers for any other explicit equations that they are familiar with and these will be included as well. The four equations that I plan to discuss are:

Serghide's Solution.³

Zigrang and Sylvester Solution.⁴

Swamee and Jain.⁵

Altshul-Tsal.⁶



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Special Cases:

In this section, several special case equations will be examined. Where possible, the deviations between these equations and the parent equations will be evaluated. We will also examine if these special case equations are needed because the Colebrook solutions are inadequate or if they are useful as merely simplified equations.

What should be considered in selecting a method to solve Colebrook:

Several alternatives will be presented for calculating the Friction Factor, "f". Easy of use, accuracy, alternatives and limits of use are among the considerations to be evaluated.

Implicit Forms of Colebrook

There are at least three forms of the Colebrook Equation that can be found in current literature on hydraulics. These are:

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{\epsilon}{3.7 D} + \frac{2.51}{R \sqrt{f}} \right) \quad \text{Eq 1}$$

$$\frac{1}{\sqrt{f}} = 1.74 - 2 \log_{10} \left(\frac{2^* \epsilon}{D} + \frac{18.7}{R \sqrt{f}} \right) \quad \text{Eq 2}$$

$$\frac{1}{\sqrt{f}} = 1.14 + 2 \log_{10} \left(\frac{D}{\epsilon} \right) - 2 \log_{10} \left(1 + \frac{9.3}{R^* \frac{\epsilon}{D} + \sqrt{f}} \right) \quad \text{Eq 3}$$

Where:

f is the Friction Factor and is dimensionless

ε is the Absolute Roughness and is in units of length

D is the Inside Diameter and, as these formulas are written, is in the same units as ε.

R is the Reynolds Number and is dimensionless.

Note that ε/D is the Relative Roughness and is dimensionless.

These three equations are referred to as "Implicit" Equations. "Implicit" means that "f", the Friction Factor, is "Implied or understood though not directly expressed". Simply stated, the equations ARE NOT in the form of "f =". These are sometimes referred to as "equivalent" but as we will see, the results will vary when calculated to the fourth significant digit.



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These equations can be solved for “f” given the Relative Roughness (ϵ/D) and the Reynolds Number, (R), by iteration. Such iterations can be performed using an electronic spreadsheet. A spreadsheet, “Friction Factor Formulas for Cheresource.xls” is presented for demonstration. The spreadsheet contains four worksheets. The first “Tab” is labeled “Iterations”. The iterative solutions are generated by breaking the formulas in two parts, that which is left of the equal sign and that which is right of the equal sign. (See row 20 as an example.) The iteration then tests values of “f” that will result in the difference between the two sides to be zero or very close to zero. (A complete explanation was published in the ASHRAE Journal of September, 2002⁷.)

If you open the spreadsheet to the “Iterations” workbook and enter values for Relative Roughness of .001 in cell C3 and Reynolds Number of 1,000,000 in cell C4, you will see cells C9, C11 and C13 all indicate that “Iteration Required”. This is indicating that a solution has not been found and that an iteration must be performed. You will note that there is a Command Button titled “Perform Iterations” in the upper left hand area of the spreadsheet. A macro was written to perform the necessary iterations on the three formulas when this Command Button is clicked. (You can alternatively execute the macro for the three iterations using the Hotkeys “Ctrl – I” or execute the iterations manually using the Tools / Goal Seek commands from the menu bar.)

Assuming that the values suggested have been entered and the Command Button has been clicked, cells C9, C11 and C13 should now contain values. Note that each value is identical for at least three significant digits, that being .0199...for the example given. (If you refer to a Moody Diagram, you will see that the graphical solution appears to be approximately .02.) You should also note that there is some difference between these cells starting at the fourth significant digit. You may wonder if this is an error in the iteration or difference in the formulas themselves.

The accuracy of the iterative solution can be validated by back substituting the result into the original equation. I’m going to refer to this as the “Check Value”. Modifying Eq 1 by multiplying both sides of the equation by the square root of “f” becomes:

$$1 = -2 \log_{10} \left(\frac{\epsilon}{3.7D} + \frac{2.51}{R \sqrt{f}} \right) * \sqrt{f}$$

Back substituting the given Relative Roughness and Reynolds Numbers and the calculated Friction Factor, an exact solution would result in a “Check Value” of:

$$1 = 1.000000.....$$



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The result of an iteration is limited by either the number of iterations or the maximum deviation allowed. The result here, although extremely accurate, is less than exact. (You may note that the difference in cell I20 was multiplied by 1000 to improve the accuracy over the default "Maximum Change" of .001 in Excel.) Given that the original formula was empirically derived, and that there is some difference between the results from the three forms of Colebrook, something less than the exact solution must be deemed acceptable. What is practical from the stand point of Good Engineering Practice is something that will be discussed in the last section of this series. For now, I will present "Check Values" taken to the sixth decimal point for comparing various solutions.

In the spreadsheet, you will notice that "Check Values" are given in both columns D and E. In column D, the "Check Values" are calculated using the same equation used in the iteration. This indicates that the iterations produced very accurate results. Column E, on the other hand, is calculated using the Friction Factor from the iteration of Eq 2 or 3, but back substituted into Eq 1 for calculating the "Check Values".

There is a problem with Eq 3; it isn't capable of producing a result when a Relative Roughness of zero (representing smooth pipe) is entered. The Visual Basic dialog box will appear with the message, "Run-time error '1004'. Reference is not valid" Click on the "End" Button to get out of the macro. A good approximation can be calculated by entering a very small Relative Roughness, say .000 000 001.

Enough for now on Iterative Solutions. We will revisit this subject and the other solutions to be covered in the last section of this series.

User Defined Functions:

As an alternative to solving Colebrook using Iteration, User Defined Functions (UDFs) can be written that use a variety of methods to solve Colebrook. In this series, we will examine the use of an iterative like approach that doesn't require the initiation of the built-in Iteration Function (Goal Seek). One advantage of this UDF that it can be used in a series of calculations such as a piping network without initiating an Iteration. Another advantage is that it can be used in conjunction with another Iteration without embedding Iterations. An example of this second advantage would be to Iterate for a fixed pressure drop by changing the flow, diameter or Relative Roughness. An example will be presented.

The UDFs presented in the demonstration spreadsheet are defined in the Visual Basic Editor. (It's not necessary to examine them but if you desire to do so, they



can be viewed from the menu bar Tools/Macro/Visual Basic Editor or from the keyboard shortcut "Alt-F11". In the Editor, the UDF "=fEq1()" is Module1. These UDFs require the bare minimum knowledge of Visual Basic. This, however, is beyond the scope of this series, although I would encourage the reader to look at how they have been created.)

Each UDF is basically identical with only the difference in the right side of the equation changed for the three Colebrook Equations. The basic routine is:

- Visualize a plot of the Difference (as described in the previous Section) versus the Friction Factor.
- Get an initial value for "f" using the Swamee and Jain equation. (More on Swamee and Jain Equation in the Explicit Section.)
- Enter a "Do Loop Until" loop.
- Calculate the slope of the Difference line.
- Project where a straight line with this slope and "f" value will cross the Friction Factor line.
- Using the projected value from above, repeat the process until the conditions are satisfied.

For a detailed explanation, refer to "Solving for Friction Factor", ASHRAE Journal July 2003⁸.

Open the workbook under the Tab "User Defined Func". In column G the corresponding UDF is given for each form of Colebrook. As different values of Relative Roughness and Reynolds Number are entered, the results from the UDFs are changed immediately. The Iteration Command Button can be clicked to perform the Iterations as before so that the Iteration results can be compared with the UDFs. "Check Values" in column I show that the UDF's have successfully calculated the Friction Factor to 1.000000 in all cases.

Example:

From the "User Defined Func" workbook, first enter a Relative Roughness in cell C3 of .005 and a Reynolds Number of 1,000,000. The various Friction Factors will be approximately .0304..... Now lets assume that we want to know what Relative Roughness will give us a Friction Factor of .0200.

- In cell E9 enter the formula "=G9*1000" without the quotation marks. (This is necessary to achieve the desired accuracy.)
- Place the cursor in cell E9.
- Select Tools\Goal Seek from the menu.
- The "Set Cell" should show "E9" if the cursor started in cell "E9"
- Tab down to the "To Value" and enter 20. (This is 1000 times the desired Friction Factor of .0200.)
- Tab down to the "By Changing cell" and enter C3. (This is the Relative Roughness number that will be changed to give the desired result.)

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-Hit the "OK" Button and again hit the "OK" Button.

At this point, the Relative Roughness in cell C3 should have changed to .0010124 and the Friction Factor in cell G9 should have changed to .0200....

UDFs exist in the spreadsheet that they were created in or in a spreadsheet were they have been copied to. They don't exist in a blank spreadsheet. To facilitate the process of using the UDF's in an existing or new spreadsheet, each UDF has been exported to a "filename".bas file. To copy a UDF to a blank or existing spreadsheet;

- Place the desired "filename".bas in a directory where it can be easily found.
- Open the spreadsheet that you wish to add the UDF to.
- Go to the Visual Basic Editor. (From the menu, Tools/Macro/Visual Basic Editor or use the keyboard shortcut "Alt-F11").
- In the Visual Basic Editor, from the menu bar, do a File/Import File/Highlight the file to be imported, a "filename".bas file.
- Hit the OK button and close the Visual Basic Editor.

The UDF is now part of your spreadsheet and will be saved as part of the file, when you save it.

To use the new spreadsheet, first decide which cells will contain the Relative Roughness and the Reynolds Number. Second, you can do either;

- From the Functions List, under User Defined Functions, you can select the Function and assign cell references for the Relative Roughness and Reynolds Number
- or
- Simply write in the formula as "=fEq1(C3,C4)" assuming you want to use "fEq1()", the Relative Roughness is in cell C3 and Reynolds Number is in cell C4.

There is a problem with Eq 3; it isn't capable of producing a result for a Relative Roughness of zero (0), (representing smooth pipe). If the Iteration is run with a zero (0) entered, the Visual Basic dialog box will appear with the message, "Runtime error '1004'. Reference is not valid" Click on the "End" Button to get out of the macro. A good approximation can be calculated by entering a very small Relative Roughness, say .000 000 001. The UDF for Eq 3 will return "#VALUE!" with zero (0) Relative Roughness.

That completes the Section on UDFs. Next we will examine several Explicit Functions to calculate Friction Factor.

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Explicit Forms

As mentioned in the Introduction, there are four Explicit Equations that will be discussed.

Serghide's Solution.³

$$A = -2 \log_{10}[(\text{RelRough} / 3.7) + (12 / \text{Reynolds\#})]$$

$$B = -2 \log_{10}[(\text{RelRough} / 3.7) + (2.51 * A / \text{Reynolds\#})]$$

$$C = -2 \log_{10}[(\text{RelRough} / 3.7) + (2.51 * B / \text{Reynolds\#})]$$

$$f = (A - ((B - A)^2 / (C - (2 * B) + A)))^{-2}$$

Serghide can be used across the entire range of the Moody Diagram. Its accuracy is unparalleled amongst the Explicit Equations evaluated here. It appears to be based on Eq 1, as do all the Explicit Equations presented. There is less deviation between Serghide and Eq 1 than there is between Eq 1 and either Eq 2 or Eq 3.

The soft spot, if one can call this minimal deviation a soft spot, exists with Smooth Pipe ($\epsilon/D = 0$) and a Reynolds Number of 170,000. At this point, the deviation between Serghide and the iterative solution of Eq 1 is 0.0031%. Because Serghide so closely mirrors Eq 1, it has approximately the same deviation to Eq 2 and Eq 3 as does Eq 1.

Serghide is perhaps the most complex entry that must be made into a spreadsheet. The A, B and C parameters can be entered into separate cells and then the Friction Factor can be calculated in a fourth cell. In the actual demonstration spreadsheet, under the Tab "Explicit Eq", the calculations are placed in cells B42 to C48. A User Defined Function, fSerg, was written as well. The VBM for this is fSerg.bas and can be copied to an existing worksheet in the same manner as described in the User Defined Function Section.



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Zigrang and Sylvester Solution.⁴

$$f = 1 / (-2\log_{10}(\text{RelRough} / 3.7 - 5.02 / \text{Reynolds\#} * \log_{10}(\text{RelRough} / 3.77 - 5.02 / \text{Reynolds\#} * \log_{10}(\text{RelRough} / 3.77 + 13 / \text{Reynolds\#}))))^2$$

Zigrang, like Serghide, can be used across the entire range of the Moody Diagram. Of the Explicit Equations evaluated here, It is second in accuracy to Serghide. The soft spot exists with Smooth Pipe ($\epsilon/D = 0$) and a Reynolds Number of 64,500. At this point, the deviation between Zigrang and the iterative solution of Eq 1 is 0.11%. (This still compares favorable with the maximum deviations between Eq 1 and either Eq 2 or Eq 3.) The deviation between Zigrang and Eq 2, at these same conditions, is 0.22%.

One significant advantage of Zigrang is that it can be placed in a single cell of a spreadsheet, albeit a long entry.

Swamee and Jain.⁵

$$f = .25 / (\log_{10}((\text{RelRough} / 3.7) + (5.74 / \text{Reynolds\#}^{.9})))^2$$

Swamee & Jain has limits but varies sources state these limits differently. The referenced source states the limits as:

$$10^{-6} < \epsilon/D < .01 \text{ and } 5000 < \text{Reynolds Number} < 3 \times 10^8$$

Statements vary around accuracy but the reference states, "An easier, and almost as accurate procedure as the Moody Diagram is to use the empirical formulas of Swamee and Jain,....." Deviation to Eq 1 of 2.8+% is seen at ϵ/D of .01 and Reynolds Number of 5000.

Swamee and Jain is easily entered into a single cell of a spreadsheet. The fact that it has a limited range of use, while other Explicit Equations, specifically Serghide or Zigrang, do not, is a significant disadvantage.

Altshul-Tsa⁶

$$f' = 0.11 * (\text{RelRough} + 68/\text{Reynolds\#})^{.25}$$

$$\text{if } f' < .018, f = .85 f' + .0028$$

$$\text{otherwise, } f = f'$$



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Altshul – Tsal can be found in numerous references and is generally not accompanied with any limited range of use. This is regrettable as its accuracy is limited to Relative Roughness in the lower half of the Moody Diagram. In the extreme case of Relative Roughness of .05, there is a 27+% deviation with the iterative solution of Eq 1, across the entire range of Reynolds Numbers.

Special Cases:

Terminology is critical when we speak of Special Cases. Literature is inconsistent in referring to the various areas of the Moody Diagram. As stated previously, I will strive to use terminology consistent with the original article of Moody (1944) and the Moody Diagram shown in that paper.

The zones and special lines in the Moody Diagram are:

Laminar Zone: This is the area of Reynolds Number less than 2000. In this zone, the Friction Factor is defined as $f = 64 / \text{Reynolds Number}$. The Colebrook Equation does not apply.

Critical Zone: This is the area between Reynolds Numbers greater than 2000 and less than 4000. The Colebrook Equation is not intended for this area.

Smooth Pipe: This is the line drawn at Relative Roughness, (ϵ/D) equal to zero.

Dashed Line: This is the line plotted from the relationship:
 $1 / f^5 = R \times \epsilon/D / 200$.

Transition Zone: Area bound by Reynolds Number greater than 4000, the Smooth Pipe Line and the Dashed Line.

Complete Turbulence, Rough Pipe: Area to the right of the Dashed Line.

By applying a set of logical “IF Statements” consistent with these definitions, we can determine where on the Moody Diagram a set of conditions lies. This is done in the demonstration spreadsheet, under the Tab “Special Cases”, cells E1 to I4. Knowing the Zone, we can determine if a “Special Case” applies.

The Special Cases evaluated here are those dealing with Complete Turbulence, Rough Pipe and Smooth Pipe. The first case is the Complete Turbulence, Rough Pipe.



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Looking at Eq 1 and a Moody Diagram, it can be seen that;

1. The fraction with Reynolds Number in denominator approaches zero as Reynolds Number becomes larger and larger.
2. At higher Relative Roughness, the Reynolds Number has less impact on the Friction Factor.
3. At Reynolds Number of 10^8 , all Relative Roughness curves are essentially flat and the Friction Factor is independent of the Reynolds Number.

This gives rise to a simplification of the Colebrook Equation where the Friction Factor is factorable:

$$f = 1 / (2 * \log_{10}(3.7 / \epsilon/D))^2$$

One question might be, "Does the Colebrook Equation still produce accurate results under the above condition?" The demonstration spreadsheet, under the Tab "Special Cases" can help answer this question. Along the Relative Roughness Curve of 0.05, examine the deviation between the Friction Factor as calculated with the full Colebrook Equation versus the Special Case:

Reynolds Number	Colebrook Eq 1	Special Case Equation
15,000*	.0730635	.0715507
1,000,000	.0715738	.0715507
100,000,000	.0715509	.0715507

* The Dashed Line intersects the ϵ/D of .05 at approximately this point

Two observations can be made:

1. At the Dashed Line, there is a definite deviation, (2.07... %)
2. At Reynolds Number of 100,000,000, the deviation is extremely small

With regard to the first observation, a visual examination of the Moody Diagram will reveal that at .05 Relative Roughness and Reynolds of 15,000, the Relative Roughness curve still has some curvature to it. Given that, some deviation should be expected.

The example used is an extreme case but it shows that the Colebrook Equation is accurate in this Special Case situation. This leads me to conclude that the Special Case Equation was developed for ease of use and not accuracy.

The second Special Case is used for Smooth Pipe, where the Relative Roughness is zero. Here again, simplification of the Colebrook Equation, for this



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condition, exists that is factorable for "f". Two equations are given for two ranges of Reynolds Numbers:

$$f = 0.3164 / \text{Reynolds Number}^{.25}$$

limited to Reynolds Number $< 10^5$

and

$$f = 0.0032 + 0.221 / \text{Reynolds Number}^{.237}$$

limited to $10^5 < \text{Reynolds Number} < 3 \cdot 10^6$

Deviations of 1.09...%, at Reynolds Number of 99,999 and 1.97...% at Reynolds Number of 100,001 compared to Eq 1 can be seen in the demonstration spreadsheet. I have no standard by which to judge the deviations. At the Reynolds Numbers used, there is a .89...% variation between the two Special Cases themselves. I suspect, without an real data, that the Colebrook Equation is perfectly adequate for this Special Case situation.

[What Should be Considered in Selecting a Method to Solve Colebrook:](#)

With the spreadsheets available today, numerous methods exist for calculating the Friction Factor from the Colebrook Equation. Some cover the entire range of the Moody Diagram while others are limited to only part of the Diagram. Special Cases, while simpler in format, are limited in their application as well. It is the writers opinion the there is an overwhelming advantage to using a method that has no limits. This seems especially true where a spreadsheet will be shared. While the original writer of the spreadsheet may be aware of its limitations, use by others, not familiar with these limitations, could lead to significant inaccuracies.

The issue of Ease of Use is very much as individual matter. Some may shy away from UDF's as too complicated but for myself, I find UDF's very easy to incorporate in both new and existing spreadsheets. Once they exist in a spreadsheet, I find them easier to enter than the Explicit Equations that I would consider as acceptable alternatives.

Iterations, once setup are easy to use on an individual case but are not easy to use in a piping network. This is particularly true when "What If" scenarios are being evaluated.

Eq 3 and fEq3 do not produce a result with Relative Roughness of 0.0. While there are ways to deal with this, such as using an extremely low Relative Roughness, they do posse a problem that isn't an issue with many other methods.



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Whats acceptable? That's up to the engineer. For me, I want a solution that covers the full range of the Moody Diagram, is as accurate as the deviations between the various forms of Colebrook and is easy to use. I find that all of the following meet these criteria:

1. UDF's "fEq1", fEq2" or "fSerg"
2. The Explicit Methods of Serghide and Zigrang

References:

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