Lecture 9/03/24

Review of Second Order Systems

1. Consider $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

DC Gain: $G(s)|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$

High Frequency Response: $G(s)|_{s\to\infty} = \frac{\omega_n^2}{\infty^2+2\zeta\omega_{n\infty}+\omega_n^2} = 0$

Therefore, G(s) → low pass response

2. Unit Step Response

Unit step function, c(t) = u(t) → $C(s) = \frac{1}{s}$, is our input signal

Output signal is r(t), also R(s)

$$R(s) = C(s)G(s) = \frac{1}{s}\left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]$$

$$r(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right) sin\left(\omega_n\sqrt{1-\zeta^2}t + \theta\right)$$

$$\theta = tan^{-1}\left(\frac{1-\zeta^2}{\zeta^2}\right)$$

r(t) has a steady state response (SR) and a transient response (TR)

Therefore, $r(t) = SR + TR = 1 + [TR\ term]|_{\zeta-dependent}$

If ζ = 0: undamped response: $r(t) = 1 - \sin(\omega_n t)$

If 0 < ζ < 1: underdamped response: r(t) is a damped sinusoid

If ζ = 1: response is critically damped: no oscillation in r(t)

If ζ > 1: response is overdamped: r(t) is a weighted sum of two exponential functions

$$r(t) = 1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2}$$

See the chart below:

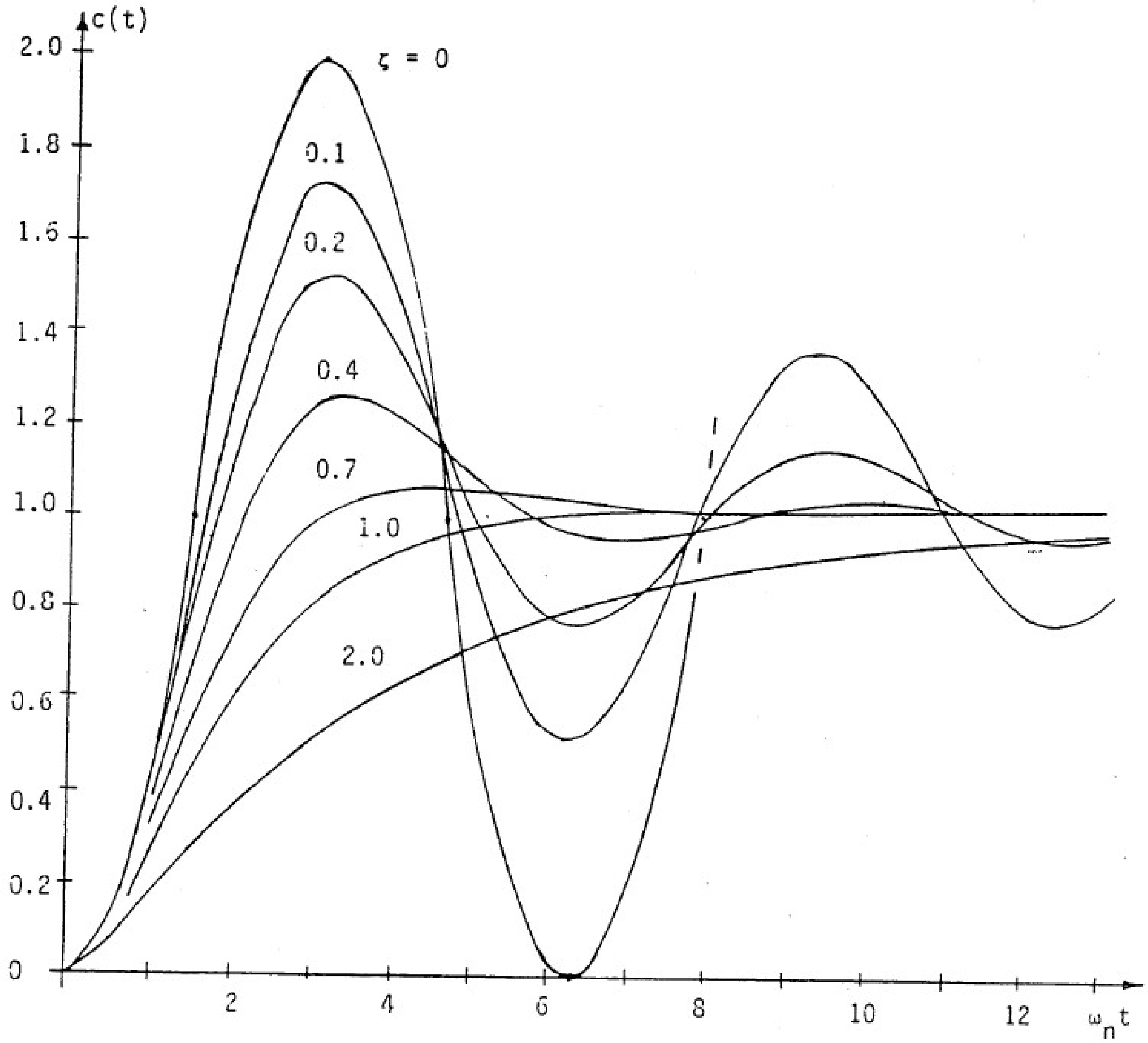


Figure 4.3 Step response for second order system (4-28).

Note: $\zeta = 0.707$ is often used in control systems, since it has a fast response time with only a little overshoot and oscillation

3. Frequency Response of G(s)

See the chart below:

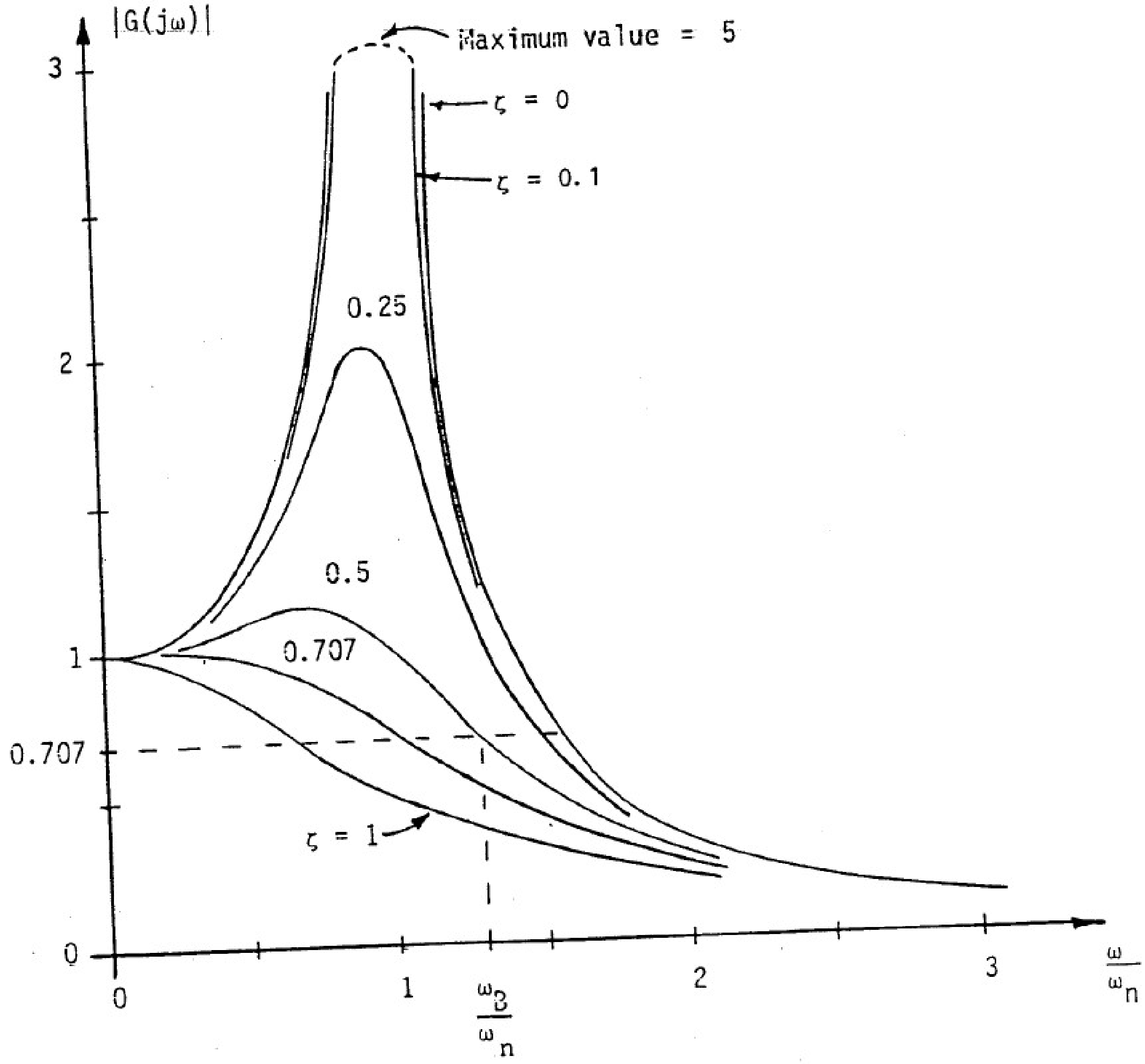


Figure 4.11 Frequency response of second order system (4-50).

Observation about the chart:

- The response is low pass
- For $\zeta < 0.707$: a resonant frequency peak occurs at ω_r, where:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

- For $\zeta > 0.707$: no resonant frequency peak occurs
- For $\zeta = 0.707$: called the Maximally Flat Response, no resonant peak occurs. The 3 dB bandwidth = ω_n.

4. Second Order System Types (Electronic Filters)

a. Low Pass Filter

$G(s) = \frac{n_0}{s^2+2\zeta\omega_n s+\omega_n^2}$ → no numerator zeros

b. High Pass Filter

$G(s) = \frac{n_2 s^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ → 2 numerator zeros at the s-plane origin

c. Band Pass Filter

$G(s) = \frac{n_1 s}{s^2+2\zeta\omega_n s+\omega_n^2}$ → 1 numerator zero at the s-plane origin

d. Notch Filter

$G(s) = \frac{n_2(s^2+\omega_n^2)}{s^2+2\zeta\omega_n s+\omega_n^2}$ → numerator zeros on the s-plane imaginary axis at $s = \pm j\omega_n$

Example filter responses:

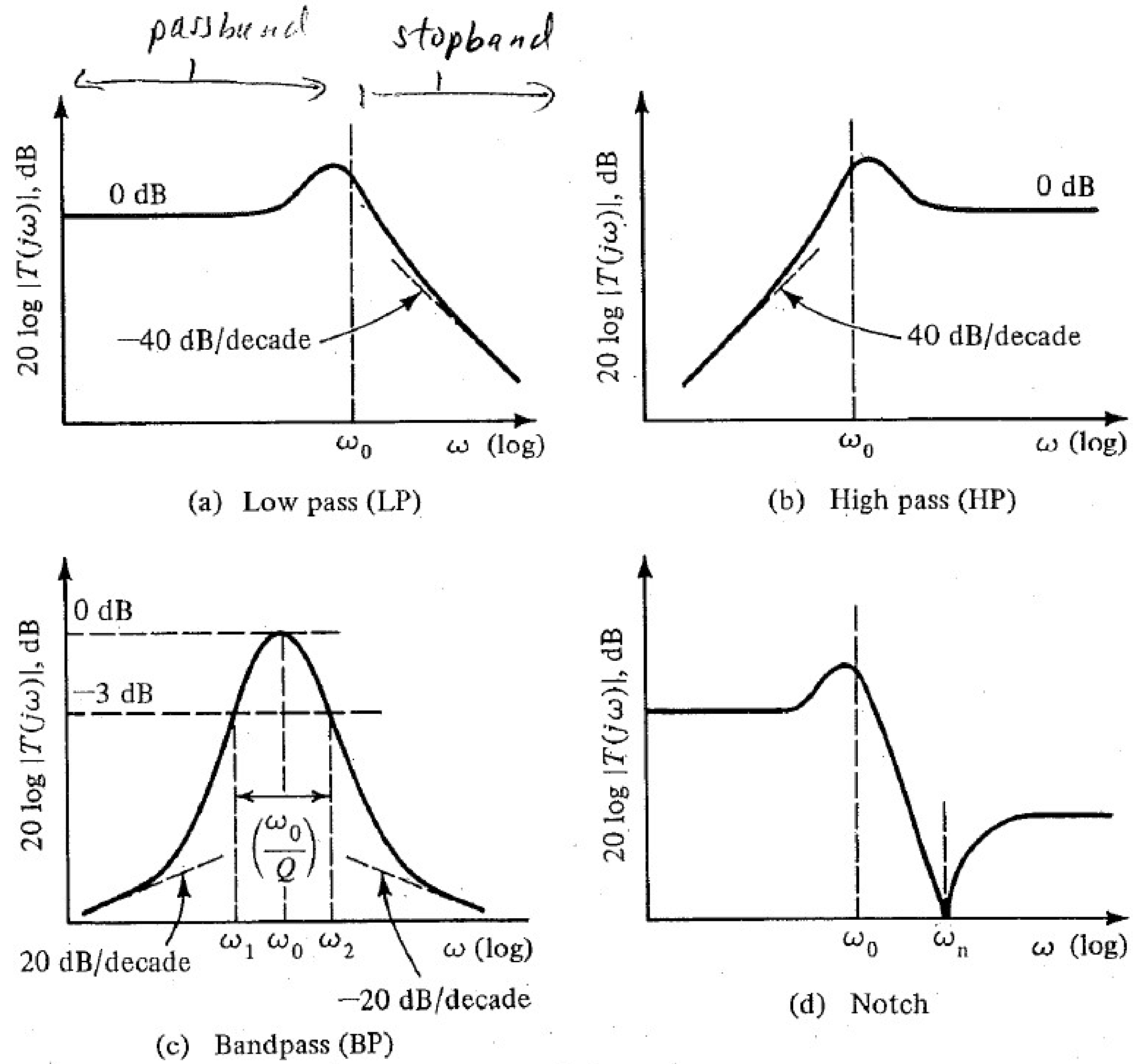


Fig. 14.2 Second-order filter responses.

System Dynamics with Damping Included

$F_I + F_D + F_S = 0 \rightarrow m\ddot{x} + c\dot{x} + kx = 0$ Note: using x or y for displacement changes nothing.

Let's rewrite the equation as: $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

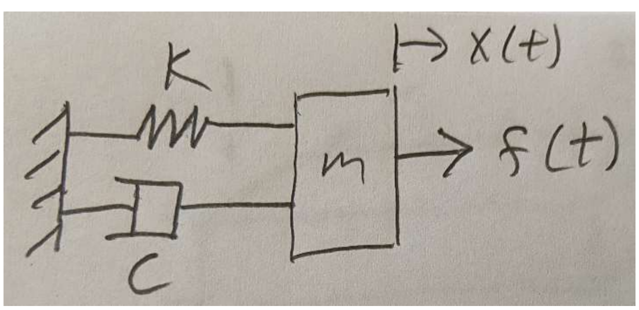
Use $\omega_n = \sqrt{\frac{k}{m}} \equiv$ natural frequency

$\zeta \equiv$ damping ratio and Q ≡ mechanical quality factor, where:

$\frac{c}{m} = 2\zeta\omega_n = \frac{\omega_n}{Q} \rightarrow Q = \frac{1}{2\zeta}$: High Q = low damping

$\therefore\ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$: another form of our system differential EQ.

Let's apply an external force, f(t), to our MEMS device:



Now: $m\ddot{x} + c\dot{x} + kx = f(t)$

Using Laplace transforms: $ms^2X(s) + csX(s) + kX(s) = F(s)$

Or: $X(s)[ms^2 + cs + k] = F(s)$

We can define a mechanical transfer function: T(s), where:

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

So what does this mean?

At DC, f(t) would be a constant force producing a displacement of the proof mass: $x = \frac{f}{k}$

At AC, f(t) is a sinusoidal force causing the proof mass to oscillate back and forth at a frequency, f, (i.e. it vibrates).

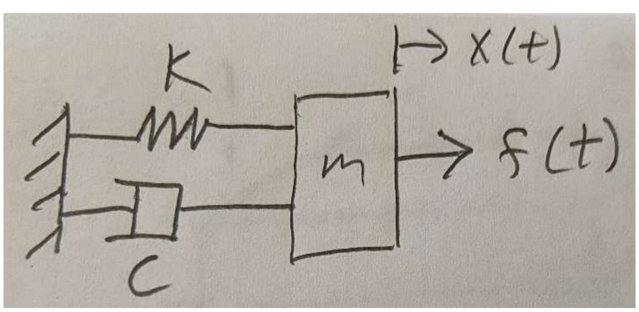
T(s) is a second order function with a low pass frequency response.

Therefore, our spring-mass-damper is a mechanical 2nd order low pass filter that filters mechanical vibrations.

If our device has very low damping (high Q), which is very common for MEMS devices, it will have a mechanical gain near f_n (small force: large proof mass motion in the vicinity of f_n).

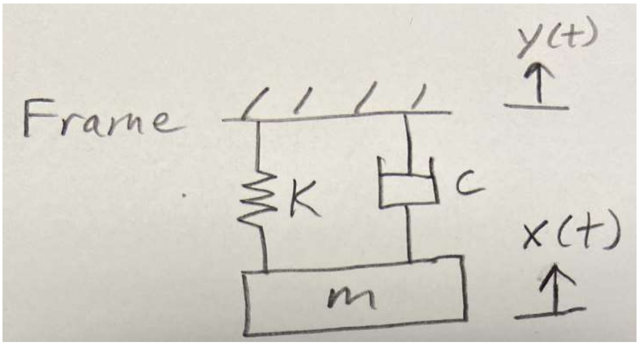
Transmissibility

Consider this model for our MEMS device:



It is often difficult to apply a specific force to the proof mass.

Therefore, consider this arrangement for our SMD system:



"SMD" = "Spring-Mass-Damper"

Define: y(t) = input displacement to the frame
x(t) = output displacement of the proof mass

It is not difficult to apply an exact displacement to the frame (outer part of the chip).

Note: $x(t) = x_o \sin(\omega t)$ → sinusoidal time varying displacement
∴ $\dot{x}(t) = \omega x_o \cos(\omega t)$ → velocity
∴ $\ddot{x}(t) = -\omega^2 x_o \sin(\omega t)$ → acceleration

Therefore, our system dynamics differential equation becomes:

$F_I + F_D + F_S = 0$

$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$ → SDEQ

Note: (x - y) > 0 → spring in compression
(x - y) < 0 → spring in tension

Take the Laplace transform of the SDEQ

$$ms^2X(s) + cs(X(s) - Y(s)) + k(X(s) - Y(s)) = 0$$

$\therefore X(s)[ms^2 + cs + k] = Y(s)[cs + k]$

Yielding: $T(s) = \frac{X(s)}{Y(s)} = \frac{cs+k}{ms^2+cs+k} = \frac{2\zeta\omega_n s+\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Observe that there is one zero in the numerator at $s = -\frac{\omega_n}{2\zeta}$, on the real axis in the s-plane. This is different from the electronic filters we discussed.

$T(j\omega) = |T(j\omega)|\,\underline{|\theta(j\omega)}$

$|T(j\omega)| \equiv$ Transmissibility

$$|T(j\omega)| = \sqrt{\frac{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} = \sqrt{\frac{1+\left(\frac{\omega}{Q\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(\frac{\omega}{Q\omega_n}\right)^2}}$$

$$|T(j\omega)|_{\omega=\omega_n} = \sqrt{Q^2-1}$$

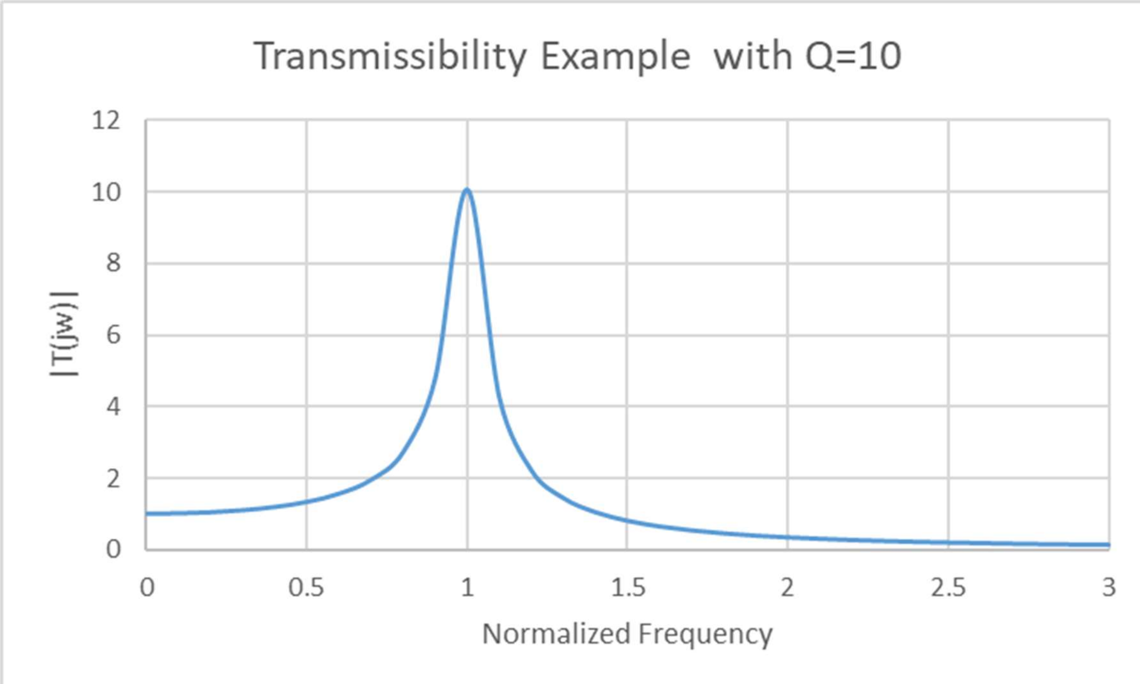
For Q >> 1 → $|T(j\omega)|_{\omega=\omega_n} \approx Q$ and $\omega_r \approx \omega_n$

Keep in mind that for Q > ½ → underdamped condition (i.e. it rings)

So, for Q >> 1, the system is highly underdamped, which is usually the case for MEMS devices.

A plot of |T(jω)| versus frequency is called a transmissibility plot.

Consider an example with Q = 10 and f_n =1 Hz (normalized to 1 Hz):



From the plot, what is Q and f_n?

What is $|T(j\omega)|$ at DC?

What is $|T(j\omega)|$ as $f \rightarrow \infty$ Hz?

$$\theta(j\omega) = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)^3}{1-\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}\right) \rightarrow \theta(j\omega)|_{\omega=\omega_n} = \tan^{-1}(Q)$$

$Q = 1 \rightarrow \theta(j\omega)|_{\omega=\omega_n} = 45^o$

$Q = 1000 \rightarrow \theta(j\omega)|_{\omega=\omega_n} = 89.94^o \rightarrow$ approaches 90^o

Also: $\theta(j\omega)|_{\omega=0} = 0^o$

And: $\theta(j\omega)|_{\omega\rightarrow\infty} = 90^o$

So what is Q? Q is a ratio of the energy stored in an oscillating system compared to the energy lost.

MEMS devices usually have a high Q (typically 25 to 1000s)

The zero in the numerator of T(s), $s = -\frac{\omega_n}{2\zeta}$, results in some interesting properties:

1) For $\omega << \omega_n$, $|T(j\omega)| \approx 1$, and $|T(j\omega)|_{\omega=0} = 1$.

2) For $\omega > \omega_n$, the stopband attenuation varies with Q:

   For Q = 1: attenuation from $2\omega_n$ to $20\omega_n$ is 21.85 dB ~ that of 1st order system.

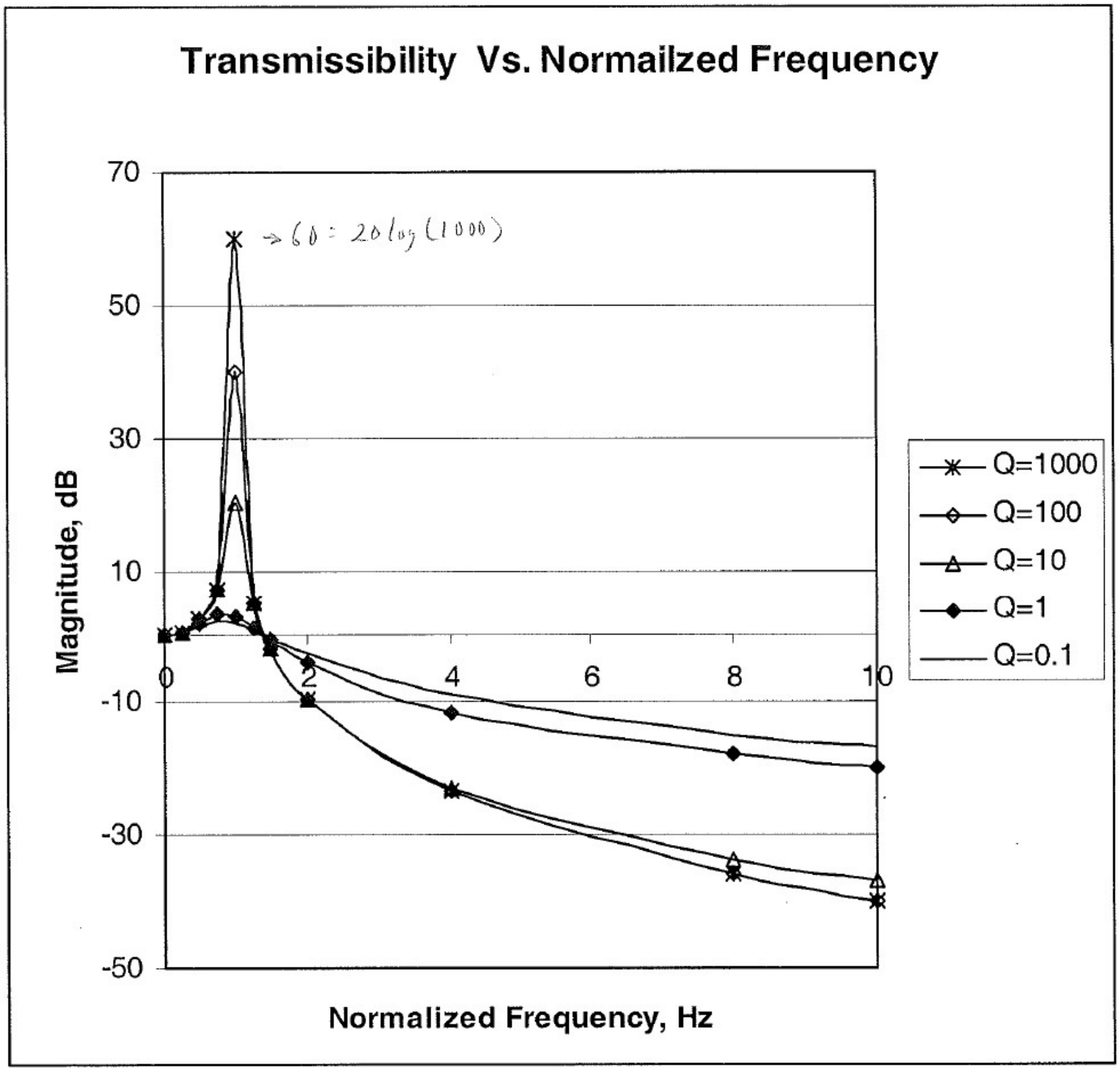
   For Q = 10: attenuation from $2\omega_n$ to $20\omega_n$ is 35.64 dB.

   For Q = 1000: attenuation from $2\omega_n$ to $20\omega_n$ is 42.48 dB ~ that of 2nd order system.

3) $|T(j\omega)| > 1$ at ω_n for any value of Q.

4) For $Q \geq 5$: $|T(j\omega)|_{\omega=\omega_n} \approx Q$. Therefore, we can read Q and f_n off of a transmissibility plot if $Q \geq 5$.

5) $[|T(j\omega)|]$ = μm/μm, m/m, etc.: i.e. dimensionless.



The Importance of Transmissibility

1. The input to a MEMS sensor might be a time dependent function of displacement, such as acceleration. Transmissibility helps us predict the device's response.

2. Microstructures can be sensitive to external mechanical noise: mechanical vibrations or acoustic energy present in the operating environment. Transmissibility reveals the susceptibility at different frequencies.

3. Often, the relative distance or velocity between the proof mass and the frame is an important parameter in a sensor: T(jω) gives us the relevant system dynamics information.

4. $|T(j\omega)|_{\omega=\omega_n}$ shows that the MEMS device performs like a mechanical amplifier at that frequency, where Q is the gain between the input displacement of the frame and the output displacement of the proof mass (for $Q \geq 5$).

Reasonable Answers with MEMS Problems

Always think about your numerical answers to see if they are reasonable.

1) Reasonable mass

Consider a "large" Si chip for a MEMS device: 1 cm x 1 cm x 500 µm.

Since $\delta_{Si} = 2.3$ g/cm^3, the entire chip can only have a mass of 115 mg.

Mass of the proof mass << 115 mg.

2) Reasonable natural frequency

$\omega_n = 2\pi f_n \rightarrow f_n$ is usually in the audio range: 20 Hz to 20 kHz.

3) Reasonable proof mass displacements

Reasonable proof mass displacements << chip width for lateral motion or << chip thickness for vertical motion: 0.1 to 10 µm is reasonable.

Note: displacements cannot exceed gap distances!

4) Reasonable capacitance values

Reasonable MEMS capacitors ~ 1 pF: 10 pF → a really large MEMS cap

Note: MEMS associated capacitances can be much smaller than this, even less than 1 fF (femto Farad: $1x10^{-15}$ F), particularly if you are considering a change in capacitance.

5) Reasonable voltages and currents

Reasonable voltages can be up to a few 100 V.

What is the approximated current from continuously fully charging and discharging a 10 pF MEMS capacitor with 100 V at 20 kHz:

Let $I = \frac{dQ}{dt} \approx fCV = 20{,}000 * 10X10^{-12} * 100 = 20\mu A$ << 1A.

Example unreasonable answers to MEMS problems:

- 1 m proof mass displacement
- 10 kg proof mass
- V = 10,000 V
- I = 10 A
- F_n = 2 MHz
- C = 10 μF