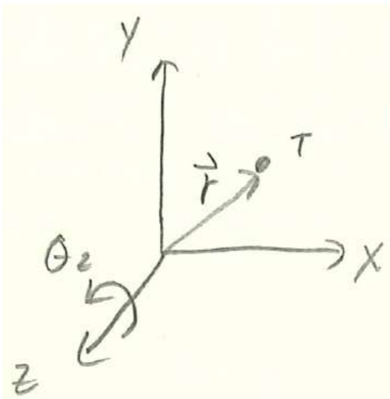
Lecture 11/5/25

Inertial Sensors (MEMS Gyroscopes)

1) Let's examine the Coriolis force and acceleration

Consider:



Where the position vector is: $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$, and

the velocity vector is: $\vec{V_r} = V_{rx}\hat{\imath} + V_{ry}\hat{\jmath} = \frac{d}{dt}\vec{r}$.

Given that the coordinate system rotates about the z-axis at: $\vec{\Omega}_z = \Omega_z\hat{k} = \dot{\theta}_z\hat{k}$, the object, T, experiences a "virtual force" in the x-y plane due the $\Omega_z\hat{k}$ rotation.

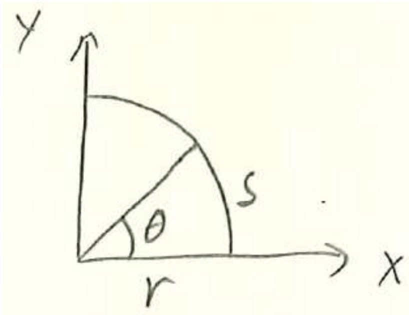
This "virtual force" is the Coriolis force, and it results in a Coriolis acceleration in the rotating x-y plane $\rightarrow \vec{a}_c$.

$$\vec{a}_c = 2\vec{\Omega}_z \times \vec{V_r} = \Omega_z\hat{k} \times \left(V_{rx}\hat{\imath} + V_{ry}\hat{\jmath}\right) = 2\Omega_z V_{rx}\hat{\jmath} - 2\Omega_z V_{ry}\hat{\imath},$$

where: $a_{cy} = 2\Omega_z V_{rx}$ and $a_{cx} = -2\Omega_z V_{ry}$.

Observe that $a_{cy} \propto \Omega_z \; multiplied \; by \; V_{rx}$.

Another way to look at this:



$S = r\theta$

$\dot{S} = \dot{r}\theta + r\dot{\theta}$

$\ddot{S} = \dot{r}\dot{\theta} + \ddot{r}\theta + r\ddot{\theta} + \dot{r}\dot{\theta}$

$= 2\dot{r}\dot{\theta} + \ddot{r}\theta + r\ddot{\theta}$

$= 2\dot{r}\Omega_z + \ddot{r}\theta + r\alpha_z$

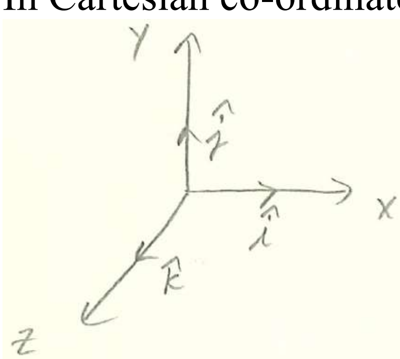
If $\theta \approx 0$ and $\ddot{\theta} \approx 0$, then $\ddot{S} \approx a_{cy} = 2V_{rx}\Omega_z$

If we were technicians, then knowing the $\vec{a}_c$ equation would be sufficient. But we are engineers and we should therefore know more.

2) Review of unit vectors

A unit vector is a normalized vector of length 1.

In Cartesian co-ordinates: 3 unit vectors in the x, y, and z directions:



For the cross product of unit vectors: use the right hand rule:

(1) $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(2) $\hat{\imath} \times \hat{\jmath} = \hat{k}$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{k} = \hat{\imath}$

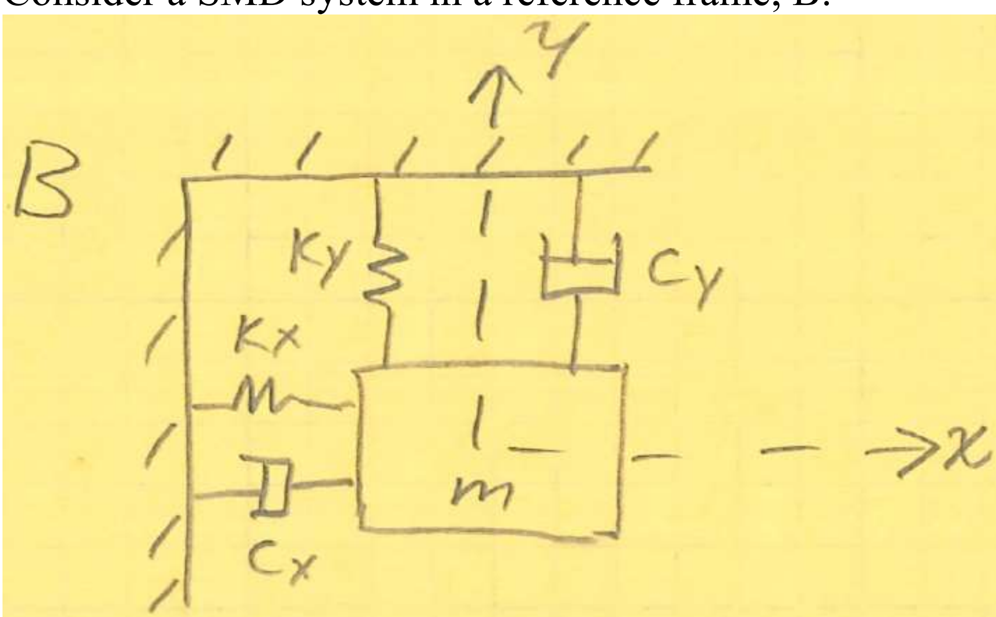
(3) $\hat{\jmath} \times \hat{\imath} = -\hat{k}$

$\hat{k} \times \hat{\jmath} = -\hat{\imath}$

$\hat{\imath} \times \hat{k} = -\hat{\jmath}$

3) Modeling an SMD in an inertial reference frame

Consider a SMD system in a reference frame, B:



Assume that the mass, m, can only move in the x and y directions with no rotation in B.

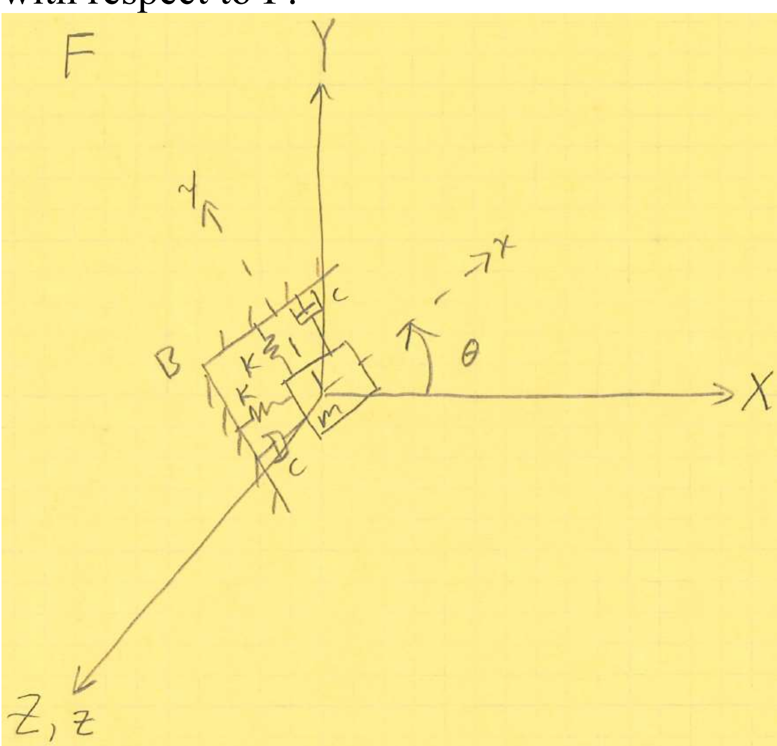
Although k_x is not necessarily the same as k_y, let's let $k_x = k_y = k$ for simplicity.

Therefore $\omega_{nx} = \omega_{ny}$ here.

Similarly, c_x is not necessarily the same as c_y, but let's let $c_x = c_y = c$ for simplicity.

4) Modeling an SMD in a rotating inertial reference frame

Let reference frame B be in a fixed reference frame F where B can rotate with respect to F:



X, Y, Z are in F → $\hat{I}, \hat{J}, \hat{K}$: unit vectors in F.

x, y, z are in B → $\hat{\imath}, \hat{\jmath}, \hat{k}$: unit vectors in B.

Note: z and Z always point in the same direction.

Let's explore the relationship between B and F:

$\hat{\imath} = f(\hat{I}, \hat{J})$ and $\hat{\jmath} = f(\hat{I}, \hat{J})$

$\therefore \hat{\imath} = \hat{I}cos(\theta) + \hat{J}sin(\theta)$

Note: if $\theta = 0^o \rightarrow \hat{\imath} = \hat{I}$

if $\theta = 90^o \rightarrow \hat{\imath} = \hat{J}$

$\therefore \hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$

Note: if $\theta = 0^o \rightarrow \hat{\jmath} = \hat{J}$

if $\theta = 90^o \rightarrow \hat{\jmath} = -\hat{I}$

Angular rate: $\dot{\theta} = \frac{d\theta}{dt} = \Omega$

Angular acceleration: $\ddot{\theta} = \frac{d\Omega}{dt} = \alpha$

5) Derivatives of unit vectors

Note: $\frac{d}{dt}(\cos(\theta)) = -\dot{\theta}\sin(\theta) = -\Omega\sin(\theta)$

And: $\frac{d}{dt}(\sin(\theta)) = \dot{\theta}\cos(\theta) = \Omega\cos(\theta)$

With that in mind:

$$\therefore\ \frac{d}{dt}(\hat{\imath}) = \frac{d}{dt}\left(\hat{I}cos(\theta) + \hat{J}sin(\theta)\right)$$

$$= -\hat{I}\Omega sin(\theta) + \hat{J}\Omega cos(\theta)$$

$$= \Omega\left(-\hat{I}sin(\theta) + \hat{J}cos(\theta)\right)$$

But: $\hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$

$$\therefore\ \frac{d}{dt}(\hat{\imath}) = \Omega\hat{\jmath}$$

$$\therefore\ \frac{d}{dt}(\hat{\jmath}) = \frac{d}{dt}\left(-\hat{I}sin(\theta) + \hat{J}cos(\theta)\right)$$

$$= -\hat{I}\Omega cos(\theta) - \hat{J}\Omega sin(\theta)$$

$$= -\Omega\left(\hat{I}cos(\theta) + \hat{J}sin(\theta)\right)$$

$$= -\Omega\hat{\imath}$$

Therefore: $\dot{\hat{\boldsymbol{\imath}}} = \Omega\hat{\boldsymbol{\jmath}}$ **and** $\dot{\hat{\boldsymbol{\jmath}}} = -\Omega\hat{\boldsymbol{\imath}}$

<u>Identities</u>

$\hat{\imath} = \hat{I}cos(\theta) + \hat{J}sin(\theta)$
$\hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$
$\dot{\hat{\imath}} = \Omega\hat{\jmath}$
$\dot{\hat{\jmath}} = -\Omega\hat{\imath}$

Corolis acceleration: $\vec{a}_c$

where $\vec{a}_c = 2\Omega\dot{x}\hat{\jmath} - 2\Omega\dot{y}\hat{\imath}$

→ motion in one axis ($\dot{x}$ or $\dot{y}$) plus rotation about z (Ω) results in motion in the opposite x or y axis:

$$\dot{x}\hat{\imath} \text{ and } \Omega\hat{k} \rightarrow 2\Omega\dot{x}\hat{\jmath}$$

$$\dot{y}\hat{\jmath} \text{ and } \Omega\hat{k} \rightarrow -2\Omega\dot{y}\hat{\imath}$$

However, higher order terms exist.

Consider the motion of the proof mass:

$\vec{r} = x\hat{\imath} + y\hat{\jmath}$ displacement of m

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + x\dot{\hat{\imath}} + y\dot{\hat{\jmath}}$$
$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \Omega(x\hat{\jmath} - y\hat{\imath}) \quad \text{velocity of m}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \dot{x}\dot{\hat{\imath}} + \dot{y}\dot{\hat{\jmath}} + \dot{\Omega}(x\hat{\jmath} - y\hat{\imath}) + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \Omega(x\dot{\hat{\jmath}} - y\dot{\hat{\imath}})$$
$$= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \alpha(x\hat{\jmath} - y\hat{\imath}) + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \Omega^2(-x\hat{\imath} - y\hat{\jmath})$$
$$= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + 2\Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \alpha(x\hat{\jmath} - y\hat{\imath}) - \Omega^2(x\hat{\imath} + y\hat{\jmath})$$ acceleration of m

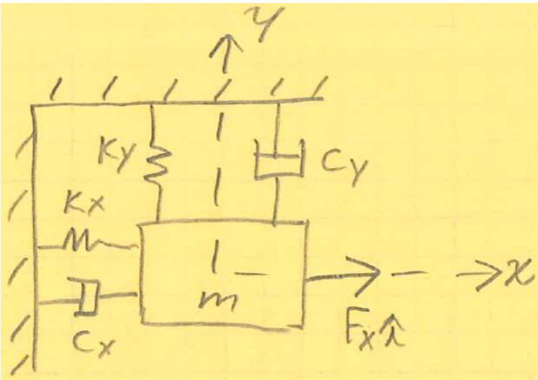
From this expression for $\vec{a}$:

$a_x = \ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x$ Acceleration component along x

$a_y = \ddot{y} + \alpha x + 2\Omega\dot{x} - \Omega^2 y$ Acceleration component along y

6) System dynamics

Consider this model for the MEMS SMD mechanical system:



$F_x = A_x \sin(\omega_d t)$ → to force m to oscillate along x–axis (using an actuator)

$F_y = 0$ → no force applied to m along y-axis (with an actuator)

There exists a coupling of the equations of motion:

$$ma_x + c_x\dot{x} + k_x x = F_x \qquad (1)$$

$$ma_y + c_y\dot{y} + k_y y = F_y = 0 \qquad (2)$$

Expanding these equations:

$$m(\ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x) + c_x\dot{x} + k_x x = A_x \sin(\omega_d t) \qquad (1)$$

$$m(\ddot{y} + \alpha x + 2\Omega\dot{x} - \Omega^2 y) + c_y\dot{y} + k_y y = 0 \qquad (2)$$

We want to solve this set of equations to obtain an expression for y(t). Thankfully, we can make some reasonable simplifying assumptions:

(1) Let $k_x = k_y = k$

(2) Let $c_x = c_y = c$

Note: with (1) and (2): $\omega_{nx} = \omega_{ny} = \omega_n$. Real MEMS gyroscopes usually have $\omega_s > \omega_d$: defined as $\omega_{ny} > \omega_{nx}$, where ω_s is in regard to the sense side and ω_d is in regard to the drive side. Having $\omega_s > \omega_d$ yields better stability and a measurable rotation rate bandwidth.

(3) Assume that the angular acceleration, α, is very slow and can be approximated as $\alpha = 0$ rad/s^2.

(4) Assume that the system natural frequency, ω_n, is much greater than Ω, the angular rate being measured. Therefore $\Omega^2 x$ and $\Omega^2 y$ can be approximated by 0.

Example: if $f_n = 10$ kHz: $\omega_n = 2\pi f_n = 62{,}831.8$ rad/s
If $\Omega = 300$ °/s = 300(2π/360) = 5.24 rad/s
And 62,831.8 >> 5.24

Also from EQ 1: $m(\ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x) + c_x\dot{x} + k_x x = A_x \sin(\omega_d t)$

Examine the "x" terms: $-m\Omega^2 x + k_x x \rightarrow m\left(\frac{k_x}{m} - \Omega^2\right) = m(\omega_n^2 - \Omega^2)$

From the Ω and f_n terms above → $\omega_n^2 = 3.9 \times 10^9$ rad/s and $\Omega^2 = 27.5$ rad/s. So, $\omega_n^2 - \Omega^2 \approx \omega_n^2$

(5) The amplitude of the motion of m along the x-axis will be tightly controlled as a closed loop resonator.

A feedback control system will adjust $F_x = A_x \sin(\omega_d t)$ to precisely keep the motion along the x-axis exactly as desired. Therefore, we can drop the $a_{cx} = -2\Omega\dot{y}$ term in EQ 1, since the controller will null out its effect.

(6) The ω_d from $F_x = A_x \sin(\omega_d t)$ is usually selected so that:

$$\omega_d = \omega_n = \sqrt{\frac{k}{m}}.$$

This minimizes the amplitude of F_x required to achieve sufficient motion of m along the x-axis → due to high Q.

Therefore, the equations of motion simply to:

$$m\ddot{x} + c\dot{x} + kx = A_x \sin(\omega_n t) \qquad (1)$$

$$m\ddot{y} + c\dot{y} + ky + 2m\Omega\dot{x} = 0 \qquad (2)$$

Clearly, $\Omega\hat{k}$ and motion along the x-axis produces corresponding motion along the y-axis {useful if $\dot{x}$ is consistent (periodic and known)}.